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June 4, 2020

On homotopy colimits

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Abstract

The goal of this thesis is to define the homotopy colimit of a functor $F\colon \mathcal{D} \to \mathcal{C}$, which we want to be a the closest, homotopy invariant, approximation to the colimit functor. We do this by introducing model categories and simplicial model categories. These are categories with additional structure making them particularly nice for doing homotopy theory in. Furthermore, we give the homotopy theoretic results needed for proving homotopy invariance of the homotopy colimit.

The definition of the homotopy colimit of F, is given as the geometric realization of a particular simplicial object, the two sided simplicial bar. Following the exposition in [Rie14] we construct this two sided simplicial bar and the bar construction. We then prove both categorical and homotopical results about this construction. Finally, we prove that if $\mathcal{M}$ is a simplicial model category, then the colimit functor $\operatorname{colim}_{\mathcal{D}}\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}$ admits a left derived functor. We then define the homotopy colimit to be $\operatorname{hocolim}_{\mathcal{D}} F \coloneqq \mathbb{L}\operatorname{colim}_{\mathcal{D}} F$.

Contents

1 Introduction

Classical category theory gives a well behaved definition of colimits. Within this framework we are able to prove that colimits are unique up to canonical isomorphism and a natural transformation between diagrams induces a unique map between the colimits of these diagrams.

However, some categories come with homotopical data of interest. One example is the category of topological spaces Top, where one often considers spaces not only up to homeomorphism but also (weak) homotopy equivalence and many of theorems in algebraic topology only concern the homotopy type of a space. This puts us at crossroads, since colimits are not necessarily homotopically well behaved. One classic counter-example is to consider the diagram

$$D^2 \longleftarrow S^1 \longrightarrow D^2$$

of spaces. The colimit of this diagram is S^2. Now if one considers the following natural transformation of diagrams

$$\begin{array}{ccccc} D^2 & \longleftarrow & S^1 & \longrightarrow & D^2 \\ \big\downarrow & & \big\downarrow \mathrm{id}_{S^1} & & \big\downarrow \\ * & \longleftarrow & S^1 & \longrightarrow & * \end{array}$$

it is well known that all the vertical arrows are homotopy equivalences. The colimit of the bottom row is the one point space $*$. However the induced map $S^2 \to *$ is not a homotopy equivalence.

In this thesis we start by presenting Quillens classical theory of model categories, which are homotopically well behaved categories. Many of the ideas presented carry over to the framework of homotopical categories. Then we turn our interest towards simplicial categories and, in order to define simplicial model categories. Then we continue to set up the framework for homotopy colimits by introducing homotopical categories, homotopical functors and derived functors. In particular we give sufficient conditions for derived functors to exist.

It turns out that simplicial model categories are categories in which we always have a good model for the left derived functor of colim. We define this to be the homotopy colimit. We will spend Section 7 proving this.

1.1 Prerequisites

This thesis is written with an introductory background in algebraic topology and homological algebra. A strong background in classical category theory is assumed - [Mac13] is a good reference for this. Additionally, [Mac13] touches on symmetric monoidal categories but only very briefly on enriched category theory. As this thesis only concerns itself with categories enriched in the category of simplicial sets $s\mathcal{S}et$ the background required is minimum, hence the presentation given in my work under supervision of Thomas Wasserman in [Nie19] suffices. Finally this thesis relies heavily on computations using coend calculus - here one classical reference besides [Mac13] is [Lor15]. We also provide an appendix on ends and coends, which provides the necessary background for the arguments given in this thesis.

1.2 Acknowledgements

I wish to extend my thanks to my supervisor Piotr Pstrągowski for his patience and guidance and will to see through this, at first, vaguely defined idea for my thesis. Additionally, I would like to thank him for encouraging my curiosity whenever it got me sidetracked during out meetings.

I also wish to extend my thanks to Thomas Wasserman for his academic guidance during most of my time as a student at UCPH. Both have truly impacted my view on mathematics.

I would also like to thanks those of my friends whose thorough reading and feedback helped me correct mistakes and writing errors.

Finally, I would like to thank the students and employees at the Math department at UCPH, whom I hope to return to after the summer. Mathematics really is best done as a group discussion, not a monologue.

Terminology and Notation

This section will introduce the basic notions used in this thesis.

Definition 1.1. Let $\mathbb{\Delta}$ denote the category of finite nonempty linearly ordered sets of the form

$$[n] = \{0, 1, \ldots, n\}$$

and order preserving maps between them. A simplicial set is an object in the functor category $s\mathcal{S}et := \mathrm{Set}^{\mathbb{\Delta}^{op}}$ and in general for a category $\mathcal{C}$ a simplicial object in $\mathcal{C}$ is an object in the functor category

$$s\mathcal{C} := \mathcal{C}^{\mathbb{\Delta}^{op}}.$$

We will always assume that any category $\mathcal{C}$ is locally small. For any pair of objects $X, Y \in \mathcal{C}$ we will denote the set of maps from X to Y by $\mathcal{C}(X, Y)$. To emphasize the difference we will denote the mapping space, from X to Y, by $\mathrm{map}(X, Y)$, when working in an simplicial category. Finally we will denote an adjunction

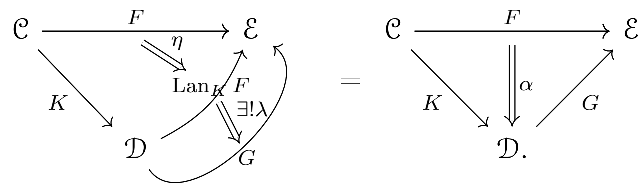
$$F \colon \mathcal{C} \leftrightarrows \colon \mathcal{D} \colon G,$$

with the convention that we always write the left adjoint on the left.

2 Kan Extensions

Definition 2.1. Let $F\colon \mathcal{C} \to \mathcal{E}$ and $K\colon \mathcal{C} \to \mathcal{D}$ be functors. A *left Kan extension of F along K* is a functor $\operatorname{Lan}_K F\colon \mathcal{D} \to \mathcal{E}$ along with a natural transformation $\eta\colon F \Rightarrow \operatorname{Lan}_K FK$ such that for any other functor $G\colon \mathcal{D} \to \mathcal{E}$ with a natural transformation $\alpha\colon F \Rightarrow GK$ there exists a unique natural transformation $\lambda\colon \operatorname{Lan}_K F \Rightarrow G$ such that $\alpha = \lambda K \eta$.

Remark 2.2. The universal property is encoded in the following equality:



Additionally it is worth mentioning that this universal property is encoded in an bijection

$$\mathcal{E}^{\mathcal{D}}\left(\operatorname{Lan}_K F, G\right) \cong \mathcal{E}^{\mathcal{C}}\left(F, GK\right).$$

If we fix K and the left Kan extension exists for all functors $F \in \mathcal{E}^{\mathcal{C}}$ this extends to an adjunction

$$\operatorname{Lan}_K\colon \mathcal{E}^{\mathcal{C}} \leftrightarrows \mathcal{E}^{\mathcal{D}} \colon K^*.$$

Theorem 2.3 ([Mac13])**.** *Suppose $\mathcal{C}$ is a small category, $\mathcal{D}$ is a category and $\mathcal{E}$ is a cocomplete category. Then the left Kan extension of any functor $F\colon \mathcal{C} \to \mathcal{E}$ along a functor $K\colon \mathcal{C} \to \mathcal{D}$ exists and can be computed at an object $d \in \mathcal{D}$ by the following formulae:*

$$\begin{aligned}\operatorname{Lan}_K F(d) &= \int^{c\in\mathcal{C}} \mathcal{D}(K(c), d) \cdot F(c) \\ &= \operatorname{colim}\left(K/d \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{E}\right) \\ &= \operatorname*{colim}_{\substack{c\in\mathcal{C} \\ K(c)\to d}} F(c)\end{aligned}$$

Here "$\cdot$" is the tensor over Set and K/d is the comma category K over $d \in \mathcal{D}$.

Remark 2.4. There are dual results for right Kan extensions. The details of these are omitted.

Definition 2.5. A functor $L\colon \mathcal{E} \to \mathcal{F}$ is said to *preserve* a left Kan extensions $(\operatorname{Lan}_K F, \eta)$ if the whiskered composition $(L \operatorname{Lan}_K F, L\eta)$ is the left Kan extension of LF along K.

Proposition 2.6. *Left adjoints preserve left Kan extensions.*

Proof. Observe that

$$\mathcal{F}^{\mathcal{D}}(L \operatorname{Lan}_K F, H) \cong \mathcal{E}^{\mathcal{D}}(\operatorname{Lan}_K F, RH) \cong \mathcal{E}^{\mathcal{C}}(F, RHK) \cong \mathcal{F}^{\mathcal{C}}(LF, HK).$$

Keeping track of $1_{L \operatorname{Lan}_K F}$ when going through these isomorphisms yields $L\eta$. □

Proposition 2.7. *Suppose $*\colon \mathcal{C} \to 1$ denotes the unique functor into the terminal category 1. Then the left Kan extension (provided it exists) of any functor $F\colon \mathcal{C} \to \mathcal{D}$ along $*$ is the colimit of F.*

Proof. We have a natural transformation $\eta\colon F \Rightarrow \operatorname{Lan}_* F*$. Let Δ denote the constant functor $\mathcal{D} \to \mathcal{D}^{\mathcal{C}}$. Suppose we have a natural transformation $\alpha\colon F \Rightarrow \Delta(x)$ for $x \in \mathcal{D}$. Then this defines a functor $G\colon 1 \to \mathcal{D}$ and a natural transformation $\alpha'\colon F \Rightarrow G*$. Then there exists a unique natural transformation $\eta\colon \operatorname{Lan}_* F \Rightarrow x$. □

Remark 2.8. Suppose $\mathcal{C}$ is small and $\mathcal{D}$ is cocomplete. Then **Theorem 2.3** implies that

$$\operatorname{colim} F \cong \operatorname{Lan}_* F(*) = \int^{c\in\mathcal{C}} 1(*(c), *) \cdot F(c) \cong \int^{c\in\mathcal{C}} \{*\} \cdot F(c) \cong \int^{c\in\mathcal{C}} F(c)$$

The above is also proven in [Mac13]. We will now prove a classical result using the machinery of coend calculus and Kan extensions.

Theorem 2.9 (Yoneda lemma)**.** *For any functor $F\colon \mathcal{C} \to \mathrm{Set}$ and $c \in \mathcal{C}$ there exists a natural isomorphism*

$$\mathrm{Set}^{\mathcal{C}}\left(\mathscr{C}(c,-), F\right) \cong F(c).$$

Proof. It is clear that $F(c) \cong \operatorname{Ran}_{1_{\mathcal{C}}} F(c)$, then by **Theorem 2.3**

$$F(c) \cong \int_{c'\in\mathcal{C}} F(c')^{\mathcal{C}(c,c')} \cong \int_{c'\in\mathcal{C}} \mathrm{Set}(\mathcal{C}(c,c'), F(c')) \cong \mathrm{Set}^{\mathcal{C}}(\mathcal{C}(c,-), F),$$

where the third isomorphism is **Proposition A.4**. □

Remark 2.10. Similarly we get that

$$F(c) \cong \int^{c'\in\mathcal{C}} \mathcal{C}(c', c) \cdot F(c').$$

This is known as the coYoneda lemma. The parameter theorem for coends [Mac13, Thm. IX.7.2] establishes that

$$F \cong \int^{c\in\mathcal{C}} \mathcal{C}(c,-) \cdot F(c)$$

which implies *the density theorem* (that all presheaves are colimits of representable presheaves). Here one needs to notice that the tensor $\cdot\colon$ Set $\times$ Set $\to$ Set is symmetric.

Theorem 2.11. *Suppose that $\mathcal{E}$ is cocomplete, $F\colon \mathcal{C} \to \mathcal{E}$ and $K\colon \mathcal{C} \to \mathcal{D}$ is fully faithful. Then*

$$\operatorname{Lan}_K F \circ K \cong F.$$

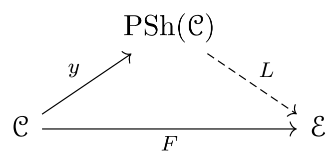
Proof. By Theorem 2.3 we see that for $d \in \mathcal{D}$

$$\operatorname{Lan}_K FK(d) = \int^{c\in\mathcal{C}} \mathcal{D}(K(c), K(d)) \cdot F(c) \cong \int^{c\in\mathcal{C}} \mathcal{C}(c,d) \cdot F(c) \cong F(d).$$

□

2.1 Defining the Geometric Realization functor

We will know study a construction which when applied to the category of simplicial sets *s*$\mathcal{S}$*et* yields a lot of useful results. Fix a small category $\mathcal{C}$ and a cocomplete category $\mathcal{E}$. We will study left Kan extensions along the Yoneda embedding $y\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$. If $F\colon \mathcal{C} \to \mathcal{E}$ is a functor, then consider $L := \operatorname{Lan}_y F$ making the diagram



commute up to natural isomorphism. Now for $P \in \mathrm{PSh}(\mathcal{C})$ we get

$$L(P) = \int^{c\in\mathcal{C}} \mathrm{PSh}(\mathcal{C})(\mathcal{C}(-,c), P) \cdot F(c) \cong \int^{c\in\mathcal{C}} P(c) \cdot F(c).$$

Additionally the functor $R\colon \mathcal{E} \to \mathrm{PSh}(\mathcal{C})$ with

$$R(e) = \mathcal{E}(F(-), e)$$

is right adjoint to L. Since if $X \in \mathrm{PSh}(\mathcal{C})$ and $e \in \mathcal{E}$ then

$$\begin{aligned}
\mathrm{Set}^{\mathcal{C}^{op}}(X, R(e)) &\cong \int_{c\in\mathcal{C}} \mathrm{Set}(X(c), R(e)(c)) \\
&\cong \int_{c\in\mathcal{C}} \mathrm{Set}(X(c), \mathcal{E}(F(c), e)) \\
&\cong \int_{c\in\mathcal{C}} \mathcal{E}(X(c)\cdot F(c), e) \\
&\cong \int_{c\in\mathcal{C}} \mathcal{E}\left(\mathrm{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-,c), X)\cdot F(c), e\right) \\
&\cong \mathcal{E}\left(\int^{c\in\mathcal{C}} \mathrm{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-,c), X)\cdot F(c), e\right) \\
&\cong \mathcal{E}(L(X), e).
\end{aligned}$$

In particular we get an adjoint equivalence of categories

$$\mathrm{Fun}(\mathcal{C}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{cocont}}(\mathrm{PSh}\,\mathcal{C}, \mathcal{E}),$$

where the right category denotes the subcategory of cocontinouos functors. Now if we specialize to the case with $\mathcal{C} = \mathbb{\Delta}$ and $\mathcal{E} = \mathrm{Top}$ we define the geometric realization to be the functor

$$|-|\colon s\mathcal{S}et \to \mathrm{Top}$$

defined as the left Kan extension of the functor $\Delta\colon \mathbb{\Delta} \to \mathrm{Top}$, with $\Delta(n) = \Delta^n$ (the standard n-simplex) along the Yoneda embedding. Then the earlier construction shows us that for some simplicial set $X_\bullet$,

$$|X_\bullet| \cong \int^{n\in\mathbb{\Delta}} X_n \cdot \Delta^n.$$

Remark 2.12. We denote the right adjoint of $|-|$ as $\mathrm{Sing}(-)$. This is easily verified to be naturally isomorphic to the usual singular set functor.

Remark 2.13. It also follows that from this construction that any presheaf category is cartesian closed.

3 Model Categories

Quillen developed the theory of *model categories* in his famous papers *Homotopical Algebra* [Qui06], to capture the notion of a good category to do homotopy theory in. In this section we will introduce the notion of model categories, and develop it sufficiently to apply the theory in this thesis. We will also give the example of the Quillen model structure on the category of simplicial sets $s\mathcal{S}et$. This is done because we intend to later define *Simplicial model categories.* Finally we give a beautiful proof of Ken Brown's lemma.

Definition 3.1. A *model category* is a category $\mathcal{M}$ equipped with three classes of maps $(W, \mathrm{cof}, \mathrm{fib})$ called weak equivalences, cofibrations and fibrations satisfying that

M1: (Limit axiom) The category is complete and cocomplete.

M2: (2-out-of-3) If $f\colon X \to Y$ and $g\colon Y \to Z$ are maps in $\mathcal{M}$ and two out of f, g or gf are weak equivalences, then so is the third.

M3: (Retract axiom) If f is a retract of g in the category of maps in $\mathcal{M}$ and g is a weak equivalence / cofibration / fibration then so is f.

M4: (Lifting axiom) If

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

is a commutative square, then l exists if

1. i is a cofibration and p is a fibration and weak equivalence.
2. i is a cofibration and a weak equivalence and p is a fibration.

M5: (Factorization axiom) Every map f factors functorially as

1. A cofibration followed by a fibration which is also a weak equivalence.
2. A cofibration which is also a weak equivalence followed by a fibration.

Definition 3.2. Let $\mathcal{M}$ be a model category. If $f\colon X \to Y$ is a map in $\mathcal{M}$ then f is a *trivial cofibration* if f is a fibration and a weak equivalence. Additionally f is a *trivial fibration* if f is a fibration and a weak equivalence.

Definition 3.3. Let $\mathcal{M}$ be a model category with $X \in \mathcal{M}$. We say that X is *cofibrant* if the unique map $\emptyset \to X$ is a cofibration. Dually X is *fibrant* if $X \to *$ is a fibration.

We will assume that the reader is familiar with classical results similar to the characterization of for instance trivial fibrations in term of lifting properties. How cofibrations are closed under coproducts and how a functorial factorization system can be used to construct a (co)fibrant replacement functor.

Remark 3.4. If $\mathcal{M}$ is a model category, then $\mathcal{M}^{op}$ is a model category in the obvious way, namely by taking $W^{op} = W$, $\mathrm{fib}^{op} = \mathrm{cof}$ and $\mathrm{cof}^{op} = fib$.

Example 3.5 (Theorem 11.1 [GJ09]). The category $s\mathcal{S}et$ has a model structure called the Quillen model structure, where a map $f\colon X \to Y$ is a

1. weak equivalence if the map between geometric realizations

$$|f|\colon |X| \to |Y|$$

is a weak homotopy equivalence of topological spaces,

2. a fibration if it is a Kan fibration, that is, if it has the right light property with respect to all horn inclusions, that is, for all n and $0 \le i \le n$ the diagram

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

admits a lift in form of the dotted arrow,

3. a cofibration if it is a monomorphism.

This following result will be usefull later.

Lemma 3.6. *If $p\colon X \to Y$ is trivial Kan fibration then p is levelwise surjective.*

Proof. Since every simplicial set is cofibrant, any lifting problem

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

has a solution. □

We will now generalize the notion of homotopy to the setting to model categories.

Definition 3.7. Let $\mathcal{M}$ be a model category and let $f, g\colon X \to Y$ be maps in $\mathcal{M}$.

1. A *cylinder* object for X is a factorization of the codiagonal map

$$X \coprod X \xrightarrow{i_0 \coprod i_1} \mathrm{Cyl}(X) \xrightarrow{p} X$$

such that $i_0 \coprod i_1$ is a cofibration and p is weak equivalence.

2. A *left homotopy* from f to g is a cylinder object on X and a map $H\colon \mathrm{Cyl}(X) \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$. We denote this $f \overset{l}{\sim} g$.

3. A *path* object for Y is a factorization of the diagonal map

$$Y \xrightarrow{s} \mathrm{Path}(Y) \xrightarrow{p_0 \times p_1} Y \times Y$$

such that s is a weak equivalence and $p_0 \times p_1$ is fibration.

4. A *right homotopy* from f to g is a path object for Y and a map $H\colon X \to \mathrm{Path}(Y)$ such that $p_0 H = f$ and $p_1 H = g$. We denote this $f \overset{r}{\sim} g$.

5. If f and g are both left and right homotopic we say that f and g are homotopic. We denote this $f \sim g$.

6. A map $f\colon X \to Y$ is a homotopy equivalence if there exists a map $g\colon Y \to X$ such that $gf \sim \mathrm{id}_X$ and $\mathrm{id}_Y \sim fg$.

Definition 3.8. The homotopy category $\mathrm{Ho}\mathcal{M}$ of a model category $\mathcal{M}$ is the formal localization of $\mathcal{M}$ at $\mathcal{W}$. That is the universal category with a functor $\mathcal{M} \to \mathrm{Ho}\mathcal{M}$ which inverts all weak equivalences.

The following theorem is a generalization of a well known theorem from algebraic topology to the setting of model categories.

Theorem 3.9 (Whitehead theorem). *Let $\mathcal{M}$ be a model category. If $X, Y \in \mathcal{M}$ are fibrant-cofibrant. Then $f\colon X \to Y$ is a weak equivalence if and only if it is a homotopy equivalence.*

Proof. This is [Hir09, Thm 7.8.5] and [Hir09, Thm. 7.5.10]. □

These results are the backbone of giving a good description of the homotopy category of a model category as the category with the same objects and homotopy classes of maps, between fibrant-cofibrant replacements of those same objects. In particular this implies that the homotopy category of a locally small model category is again locally small.

Theorem 3.10 (Ken Brown's lemma). *Let $\mathcal{M}, \mathcal{N}$ be model categories. If $F\colon \mathcal{M} \to \mathcal{N}$ is a functor that takes trivial cofibrations between cofibrant object in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$, then F preserves weak equivalences between cofibrant objects.*

Proof. Let $f\colon A \to B$ be a weak equivalence between cofibrant objects. Since the class of cofibrations are closed under pushouts the inclusions into the coproduct are cofibrations. Now factor the map

$$A \coprod B \xrightarrow{f \coprod \mathrm{id}_B} B$$

into a cofibration $A \coprod B \xrightarrow{q} C$ followed by a trivial fibration $C \xrightarrow{p} B$. Then qi_A and qi_B are cofibrations. In fact these are trivial cofibrations since $p(qi_A) = f$ and $p(qi_B) = \mathrm{id}_B$ are weak equivalances then this follows by the 2-out-of-3 property of weak equivalences. Thus $F(qi_A)$ and $F(qi_B)$ are weak equivalences in $\mathcal{N}$. Now $\mathrm{id}_{F(B)} = F(\mathrm{id}_B) = F(pqi_B)$ implies that $F(p)$ is a weak equivalence by 2-out-of-3. Therefor, 2-out-of-3 implies that $F(pqi_A) = F(f)$ is a weak equivalence. □

Theorem 3.11. *Let $\mathcal{M}, \mathcal{N}$ be model categories. If $F\colon \mathcal{M} \leftrightarrows \mathcal{N}\colon G$ is an adjunction then F preserves cofibrations and trivial cofibrations if and only if G preserves fibrations and trivial fibrations.*

Proof. We will show that F preserves cofibrations if and only if G preserves trivial fibrations. Suppose $j\colon X \to Y$ is a cofibration in $\mathcal{M}$ then $F(j)$ is a cofibration in $\mathcal{N}$ if and only if $F(j)$ has the left lifting property with respect to all trivial fibrations q in $\mathcal{N}$. Since F and G are adjoint this is true if and only j has the left lifting property with respect to $G(q)$. This is the case if and only if $G(q)$ is a trivial fibration. □

4 Simplicial Model Categories

In order to construct the model for homotopy colimits given in this thesis we need the additional structure of a simplicial model structure on $\mathcal{M}$. We therefor start by discussing simplicial categories and then turn our attention to simplicial model categories.

4.1 Simplicial categories

Recall that a simplicially enriched category (simplicial category for short) is a category enriched in $(s\mathcal{S}et, \times, *)$. It is easy to see that the data of a simplicial category $\mathcal{M}$ is equivalent to a category $\mathcal{M}$ satisfying that for each pair of objects $X, Y \in \mathcal{M}$ there exist a simplicial set $\operatorname{map}(X, Y)$ and maps of simplicial sets

$$\operatorname{map}(Y, Z) \times \operatorname{map}(X, Y) \to \operatorname{map}(X, Z), \quad * \to \operatorname{map}(X, X)$$

satisfying associativity and unitality in the appropriate sense such that $\operatorname{map}(X, Y)_0 \cong \operatorname{Hom}_{\mathcal{M}}(X, Y)$. This isomorphism should respect composition.

Theorem 4.1. *If $\mathcal{M}$ is a cocomplete category then $s\mathcal{M}$ can be simplicially enriched and tensored in a canonical manner. If additionally $\mathcal{M}$ is complete then $\mathcal{M}$ is also cotensored over $s\mathcal{S}et$ and for any simplicial set $K \in s\mathcal{S}et$ we obtain an adjunction*

$$K \otimes (-) \colon s\mathcal{M} \leftrightarrows s\mathcal{M} \colon (-)^K$$

Proof. For $K \in s\mathcal{S}et$ and $X \in s\mathcal{M}$ we define the simplicial tensor $K \otimes X$ on n-simplicies as

$$(K \otimes X)_n := K_n \cdot X_n$$

with the obvious functoriality in $[n] \in \mathbb{\Delta}^{op}$. We use this to define $\operatorname{map}(X, Y)$

$$\operatorname{map}(X, Y)_n := s\mathcal{M}(\Delta^n \otimes X, Y),$$

with face and degeneracy maps induced by the maps in $\mathbb{\Delta}$. One easily checks that this defines a simplicial structure.

We will now prove that $s\mathcal{M}$ is tensored with respect to these constructions. We will proceed by proving this at the zeroth level and then reducing the general case to this. The following isomorphisms, where we make use of the density theorem, establish the first part:

$$\begin{aligned}
s\mathcal{M}(K \otimes X, Y) &\cong s\mathcal{M}\left((\operatorname*{colim}_{\Delta^n \to K} \Delta^n) \otimes X, Y \right) \\
&\cong s\mathcal{M}(\operatorname*{colim}_{\Delta^n \to K} (\Delta^n \otimes X), Y) \\
&\cong \lim_{\Delta^n \to K} s\mathcal{M}(\Delta^n \otimes X, Y) \\
&\cong \lim_{\Delta^n \to K} \operatorname{map}(X, Y)_n \\
&\cong \lim_{\Delta^n \to K} s\mathcal{S}et(\Delta^n, \operatorname{map}(X, Y)) \\
&\cong s\mathcal{S}et(\operatorname*{colim}_{\Delta^n \to K} \Delta^n, \operatorname{map}(X, Y)) \\
&\cong s\mathcal{S}et(K, \operatorname{map}(X, Y)).
\end{aligned}$$

We now show that the n-simplicies of $\operatorname{map}(K \otimes X, Y)$ and $\operatorname{map}(K, \operatorname{map}(X, Y))$ coincide.

$$\begin{aligned}
\operatorname{map}(K \otimes X, Y)_n &\cong s\mathcal{S}et(\Delta^n, \operatorname{map}(K \otimes X, Y)) \\
&\cong s\mathcal{M}(\Delta^n \otimes (K \otimes X), Y) \\
&\cong s\mathcal{M}((\Delta^n \times K) \otimes X, Y) \\
&\cong s\mathcal{S}et(\Delta^n \times K, \operatorname{map}(X, Y)) \\
&\cong s\mathcal{S}et(\Delta^n, \operatorname{map}(K, \operatorname{map}(X, Y))) \\
&\cong \operatorname{map}(K, \operatorname{map}(X, Y))_n
\end{aligned}$$

Now, suppose $\mathcal{M}$ is also complete, so that $\mathcal{M}$ is cotensored over Set with $M^S = \prod_S M$. Let $(X^K)_n := X_n^{K_n}$ one argues that this defines a cotensoring in an argument dual to the above. To see the adjunction consider the isomorphisms

$$\begin{aligned} s\mathcal{M}(X, Y^K) &\cong \int_{n \in \mathbb{\Delta}^{op}} \mathcal{M}(X_n, (Y^K)_n) \\ &\cong \int_{n \in \mathbb{\Delta}^{op}} \mathcal{M}\left(X_n, \prod_{K_n} Y_n\right) \\ &\cong \int_{n \in \mathbb{\Delta}^{op}} \mathcal{M}\left(\coprod_{K_n} X_n, Y_n\right) \\ &\cong \int_{n \in \mathbb{\Delta}^{op}} \mathcal{M}\left((K \otimes X)_n, Y_n\right) \\ &\cong s\mathcal{M}\left(K \otimes X, Y\right) \end{aligned}$$

□

To make the exposition more clear we did not show that $- \otimes X \colon s\mathcal{S}et \to s\mathcal{M}$ commutes with colimits or that for simplicial sets $K, K' \in s\mathcal{S}et$ it holds that

$$K \otimes (K' \otimes X) \cong (K \times K') \otimes X$$

for any simplicial object $X \in s\mathcal{M}$. We will prove this now.

Lemma 4.2. *The functor $- \otimes X \colon s\mathcal{S}et \to s\mathcal{M}$ commutes with colimits for all $X \in s\mathcal{M}$.*

Proof. Since colimits in $s\mathcal{M}$ is computed levelwise, it is sufficient to see that $- \otimes X$ commutes with colimits for each $[n] \in \mathbb{\Delta}^{op}$. For a diagram

$$F \colon \mathcal{D} \to s\mathcal{S}et,$$

we see that

$$\begin{aligned} \mathcal{M}(((\operatorname*{colim}_{d \in \mathcal{D}} F(d)) \otimes X)_n, Y) &= \mathcal{M}\left((\operatorname*{colim}_{d \in \mathcal{D}} F(d)) \cdot X_n, Y\right) \\ &\cong \mathrm{Set}(\operatorname*{colim}_{d \in \mathcal{D}} F(d), \mathcal{M}(X_n, Y)) \\ &\cong \lim_{d \in \mathcal{D}} \mathrm{Set}(F(d), \mathcal{M}(X_n, Y)) \\ &\cong \lim_{d \in \mathcal{D}} \mathcal{M}(F(d) \cdot X_n, Y) \\ &\cong \mathcal{M}(\operatorname*{colim}_{d \in \mathcal{D}}(F(d) \otimes X)_n, Y) \end{aligned}$$

completing the proof. □

Lemma 4.3. *For simplicial sets $K, K' \in s\mathcal{S}et$ and a simplicial object $X \in s\mathcal{M}$ we have*

$$K \otimes \left(K' \otimes X\right) \cong \left(K \times K'\right) \otimes X.$$

Proof. This is clear from the definitions. □

Remark 4.4. When $\mathcal{M} = s\mathcal{S}et$ this recovers the usual enrichment of $s\mathcal{S}et$ and gives a construction of the internal hom of $s\mathcal{S}et$.

Definition 4.5. A *simplicial model category* is a simplicially enriched category $\mathcal{M}$ that is also a model category and such that

SM1 $\mathcal{M}$ is tensored and cotensored in simplicial sets.

SM2 If $i\colon A \to B$ is a cofibration in $\mathcal{M}$ and $p\colon X \to Y$ is a fibration then

$$\mathrm{map}(B, X) \xrightarrow{i^* \times p_*} \mathrm{map}(A, X) \times_{\mathrm{map}(A,Y)} \mathrm{map}(B, Y)$$

is a Kan fibration, which moreover is trivial whenever p or i is a weak equivalence.

Example 4.6. The category of simplicial sets $s\mathcal{S}et$ with the Quillen model structure is a simplicial model category, see [GJ09, Theorem 11.5] and **Remark 2.13**

Remark 4.7. Note that SM: 2 implies that $\mathrm{map}(A, -)$ preserves (trivial) fibrations for any fibrant object and $\mathrm{map}(-, B)$ takes (trivial) cofibrations to (trivial) fibrations if B is cofibrant.

Proposition 4.8. *Let $\mathcal{M}$ be a simplicial model category. If $K \in s\mathcal{S}et$ and $i\colon A \to B$ in $\mathcal{M}$ is a cofibration then $K \otimes i\colon K \otimes A \to K \otimes B$ is a cofibration.*

Proof. Let $p\colon X \to Y$ be a trivial fibration, by (SM2)

$$\mathrm{map}(B, X) \xrightarrow{i^* \times p_*} \mathrm{map}(A, X) \times_{\mathrm{map}(A,Y)} \mathrm{map}(B, Y)$$

is trivial Kan fibration. Since K is cofibrant

$$\mathrm{map}(K, \mathrm{map}(B, X)) \xrightarrow{(i^* \times p_*)_*} \mathrm{map}(K, \mathrm{map}(A, X) \times_{\mathrm{map}(A,Y)} \mathrm{map}(B, Y))$$

is a trivial Kan fibration. Since $\mathcal{M}$ is tensored we have a natural isomorphism

$$\mathrm{map}(K \otimes X, Y) \cong \mathrm{map}(K, \mathrm{map}(X, Y))$$

Applying this we obtain isomorphisms such that

$$\begin{array}{ccc} \mathrm{map}(K, \mathrm{map}(B, X)) & \xrightarrow{(i^* \times p_*)_*} & \mathrm{map}(K, \mathrm{map}(A, X) \times_{\mathrm{map}(A,Y)} \mathrm{map}(B, Y)) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{map}(K \otimes B, X) & \xrightarrow{(K \otimes i)^* \times p_*} & \mathrm{map}(K \otimes A, X) \times_{\mathrm{map}(K \otimes A,Y)} \mathrm{map}(K \otimes B, Y) \end{array}$$

commutes. Thus $(K \otimes i)^* \times p_*$ is trivial Kan fibration, which by **Lemma 3.6** is surjective on 0-simplicies. This shows that $K \otimes i$ has the left lifting property with respect to any trivial fibration p, hence $K \otimes i$ is a cofibration. $\square$

4.2 Geometric realization and singular complex

The goal this section is to introduce the geometric realization of simplicial object $X \in s\mathcal{M}$ in a simplicial category which is tensored. Furthere more if $\mathcal{M}$ is also cotensored over simplicial sets we will construct an right adjoint the geometric realization functor, which we will call the singular complex.

Definition 4.9. Let $\mathcal{M}$ be a simplicially enriched category, which is tensored and cocomplete. If $X_\bullet \in s\mathcal{M}$ is a simplicial object in $\mathcal{M}$, then the *geometric realization* of X is defined to be the coend

$$|X_\bullet| := \int^{n \in \Delta} \Delta^n \otimes X_n,$$

where $\Delta^\bullet$ is the standard cosimplicial object given by the Yoneda embedding $\Delta \overset{y}{\hookrightarrow} s\mathcal{S}et$.

Applying the parameter theorem for coends, see [Mac13, Thm. IX.7.2], this assembles into a functor

$$|-|\colon \mathcal{M}^{\Delta^{op}} \to \mathcal{M}$$

Definition 4.10. If $\mathcal{M}$ is simplicially enriched and cotensored we define the *singular complex* on $Y \in \mathcal{M}$ to be the simplicial object $Y^{\Delta^\bullet}$ with $\left(Y^{\Delta^\bullet}\right)_n = Y^{\Delta^n}$.

This naturally assembles into a functor $(-)^{\Delta^\bullet}$.

Proposition 4.11. *If $\mathcal{M}$ is a simplicially enriched category which is tensored and cotensored, then we have an adjunction*

$$|-|\colon s\mathcal{M} \leftrightarrows \mathcal{M}\colon (-)^{\Delta^\bullet}.$$

Proof. If $X \in s\mathcal{M}$ and $Y \in \mathcal{M}$, then

$$\begin{aligned}
\mathcal{M}(|X|, Y) &= \mathcal{M}\left(\int^{n\in\mathbb{\Delta}} \Delta^n \otimes X_n, Y\right) \\
&\cong \int_{n\in\mathbb{\Delta}} \mathcal{M}(\Delta^n \otimes X_n, Y) \\
&\cong \int_{n\in\mathbb{\Delta}} \mathcal{M}(X_n, Y^{\Delta^n}) \\
&\cong s\mathcal{M}(X, Y^{\Delta^\bullet}),
\end{aligned}$$

which shows the adjunction. □

4.3 Reedy model structure

Given a model category $\mathcal{M}$ and a small category $\mathcal{D}$ we would like to extend the homotopy theory of $\mathcal{M}$ to $\mathcal{M}^{\mathcal{D}}$, such that the weak equivalences of diagrams will be those natural transformations which are levelwise a weak equivalence. This however is in general difficult, but can always be done when $\mathcal{D}$ has the additional structure of being Reedy (for more discussion see [GJ09, Ch.VII]). The Reedy model structure will not be discussed in full, we do however need to define the Reedy model structure on $s\mathcal{M}$ and prove homotopical properties of the geometric realization functor. For this section we fix a model category $\mathcal{M}$.

Definition 4.12. Let $X \in s\mathcal{M}$ we define the *n-th latching object* of X to be the object defined by the colimit:

$$L_n X \coloneqq \operatorname*{colim}_{[n]\to[k]} X_k,$$

where the indexing category is the opposite category, of the subcategory of $[n]/\mathbb{\Delta}$, with surjective maps under $[n]$ and such that $k < n$.

Remark 4.13. When X is a simplicial set, then one can show that that $L_n X \cong (\mathrm{Sk}_{n-1} X)_n$. The n-th level of the $n-1$-skeletal approximation of X. Which turns out to be the set of degenerate n-simplicies of X.

Definition 4.14. For $X \in s\mathcal{M}$ define the *n-th matching object* of X as

$$M_n X \coloneqq \lim_{[k]\to[n]} X_k,$$

where the indexing category is the subcategory of $\mathbb{\Delta}/[n]$ with $k < n$

Remark 4.15. This turns out to be exactly $(\mathrm{cosk}_{n-1} X)_n$.

This now makes us able to state the Reedy model structure on $s\mathcal{M}$.

Definition 4.16 (VII Def. 2.1 [GJ09])**.** A map $f\colon X \to Y$ in $s\mathcal{M}$ is

1. a Reedy weak equivalence if $f_n\colon X_n \to Y_n$ is a weak equivalence for all $n \geq 0$,

2. a Reedy fibration if
$$X_n \to Y_n \times_{M_n Y} M_n X$$
is a fibration for all $n \geq 0$,
3. a Reedy cofibration if
$$X_n \cup_{L_n X} L_n Y \to Y_n$$
is a cofibration for all $n \geq 0$.

we call this structure the *Reedy structure* on $s\mathcal{M}$.

The culmination of [GJ09, VII Section 2] is the following theorem:

Theorem 4.17 (VII Thm. 2.11 [GJ09]). *If $\mathcal{M}$ is a model category, then the Reedy structure defines a model category structure on $s\mathcal{M}$.*

We will omit the proof of this theorem, since we are primarily interested in homotopical properties of the geometric realization functor and not of the homotopy theory of $s\mathcal{M}$. The following two lemmata will characterize trivial Reedy (co)fibrations and give a sufficient condition for an object to be (co)fibrant.

Lemma 4.18 (VII Lemma 2.2 [GJ09]). *A map $f\colon X \to Y$ in $s\mathcal{M}$ is*

1. *a trivial Reedy fibration if and only if*
$$X_n \to Y_n \times_{M_n Y} M_n X$$
is a trivial fibration for all $n \geq 0$
2. *a trivial Reedy cofibration if and only if*
$$X_n \cup_{L_n X} L_n Y \to Y_n$$
is a trivial cofibration for all $n \geq 0$.

Remark 4.19. Observe that an object $X \in s\mathcal{M}$ is Reedy cofibrant if
$$L_n X \to X_n$$
is a cofibration for all $n \geq 0$. Dually X is Reedy fibrant if
$$X_n \to M_n X$$
is a fibration for all $n \geq 0$.

In order to prove that geometric realization is left Quillen with respect to the Reedy structure on $s\mathcal{M}$, we will define the generalized matching of a simplicial set $K \in s\mathcal{S}et$ with a simplicial object $X \in s\mathcal{M}$.

Proposition 4.20 (Prop. VII 1.21 [GJ09]). *For a simplicial set $K \in s\mathcal{S}et$ the functor*
$$K \otimes -\colon \mathcal{M} \to s\mathcal{M}$$
given by taking the constant simplicial object and then tensoring with K admits a right adjoint.

Proof. Recall that a constant simplicial set $A \in s\mathcal{S}et$ then $s\mathcal{S}et(\Delta^n \times A, Y) \cong \mathrm{Set}(A, Y_n)$. This fact is easily generalized to our situation (for instance by applying coend calculus and the coYoneda lemma) such that for a constant simplicial object $Z \in s\mathcal{M}$ we get

$$s\mathcal{M}(\Delta^n \otimes Z, X) \cong \mathcal{M}(Z, X_n)$$

for all $X \in s\mathcal{M}$.

Now by the following string of isomorphisms we get:

$$\begin{aligned} s\mathcal{M}(K \otimes Z, X) &\cong \lim_{\Delta^n \to K} s\mathcal{M}(\Delta^n \otimes Z, X) \\ &\cong \lim_{\Delta^n \to K} \mathcal{M}(Z, X_n) \\ &= \mathcal{M}(Z, \lim_{\Delta^n \to K} X_n) \end{aligned}$$

□

Let $M_K X \coloneqq \lim_{\Delta^n \to K} X_n$ denote the right adjoint.

Remark 4.21. Note that M_K- is given by a Kan extension formula and in particular this definition coincides with **Definition 4.14** in the case where $K = \partial\Delta^n$. Furthermore for $K = \Delta^n$ it is easy to see that $M_{\Delta^n} X \cong X_n$.

We will now move towards proving that geometric realization is left Quillen. We start by proving the dual statement that the singular complex is right Quillen.

Lemma 4.22 (Lemma VII 3.17 [GJ09])**.** *The singular complex functor $(-)^{\Delta^\bullet} \colon \mathcal{M} \to s\mathcal{M}$ preserves fibrations and trivial fibrations.*

Proof. We first show that $M_K Y^{\Delta^\bullet} \cong Y^K$. This follows by the following string of isomorphisms

$$\begin{aligned} M_K Y^{\Delta^\bullet} &\cong \lim_{\Delta^n \to K} \left(Y^{\Delta^\bullet}\right)_n \\ &\cong \lim_{\Delta^n \to K} Y^{\Delta^n} \\ &\cong Y^{\operatorname{colim}_{\Delta^n \to K} \Delta^n} \\ &\cong Y^K, \end{aligned}$$

where the last isomorphism is the coYoneda lemma, and the one before that follows from adjointness. Let $f \colon X \to Y$ be a (trivial) fibration, then for all $n \geq 0$ there is a commutative diagram

$$\begin{array}{ccc} X^{\Delta^n} & \longrightarrow & Y^{\Delta^n} \times_{M_n Y^{\Delta^\bullet}} M_n X^{\Delta^\bullet} \\ \downarrow & & \downarrow \\ X^{\Delta^n} & \longrightarrow & Y^{\Delta^n} \times_{Y^{\partial\Delta^n}} X^{\partial\Delta^n} \end{array}$$

with the vertical maps isomorphisms. But then it follows from the SM2 that $f^{\Delta^\bullet}$ is a (trivial) fibration. □

Corrolary 4.23. *The geometric realization functor $|-| \colon s\mathcal{M} \to M$ is left Quillen.*

Proof. This follows from **Lemma 4.22** and **Theorem 3.11**. □

Note that in particular that by application of Ken Brown's lemma (that is **Theorem 3.10**) this means that geometric realization preserves weak equivalences between Reedy cofibrant objects.

5 Derived functors

In this section, we define homotopical categories and derived functors between them.

Definition 5.1. A *homotopical category* is a category $\mathcal{C}$ equipped with a class of maps $\mathcal{W}$ such that

1. All isomorphisms of $\mathcal{C}$ are in $\mathcal{W}$.
2. If f, g are composable arrows such that two of f, g, gf are in $\mathcal{W}$, then so is the third.

Remark 5.2. The first property is the same as saying $\mathcal{W}$ is a wide subcategory. The second property is called the *2-of-3 property*. Furthermore the maps in $\mathcal{W}$ are called weak equivalences.

Remark 5.3. Any category can be considered as a homotopical category with the subcategory $\mathcal{W}$ chosen to be minimal only containing the isomorphisms.

Example 5.4. For any homotopical category $\mathcal{C}$ and small category $\mathcal{D}$ the category $\mathcal{C}^{\mathcal{D}}$ can be considered a homotopical category with weak equivalences given by those natural transformation in which all the components are natural transformations in $\mathcal{C}$. We call such a weak equivalence a natural weak equivalence.

Example 5.5. Our main example will be a (simplicial) model category where we consider the weak equivalences as well the weak equivalences.

Definition 5.6. Suppose $\mathcal{C}$ is a homotopical category. Then the homotopy category of $\mathcal{C}$, denoted $\operatorname{Ho}\mathcal{C}$ is the formal localization of $\mathcal{C}$ at $\mathcal{W}$.[1]

Remark 5.7. The category $\operatorname{Ho}\mathcal{C}$ comes equipped with a canonical localization functor

$$\gamma\colon \mathcal{C} \to \operatorname{Ho}\mathcal{C}$$

Such that we obtain an equivalence

$$\operatorname{Fun}(\operatorname{Ho}\mathcal{C}, \mathcal{D}) \simeq \operatorname{Fun}^{\mathrm{wti}}(\mathcal{C}, \mathcal{D})$$

Where $\operatorname{Fun}^{\mathrm{wti}}(-,-)$ denotes the full sub category of $\operatorname{Fun}(-,-)$ spanned by functors sending weak equivalences to isomorphisms.

Remark 5.8. Normally one should be quite careful when forming localizations of categories as one could easily run into size issues. For model categories Quillen proves that the homotopy category of a locally small category is always locally small.

Functors which respect this extra structure are of special interest and for the rest of this section we will discuss these functors and how to approximate functors by such functors.

Definition 5.9. Let $\mathcal{C}, \mathcal{D}$ be homotopical categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called homotopical if for all $f \in \mathcal{W}_{\mathcal{C}}$ then $F(f) \in \mathcal{W}_{\mathcal{D}}$.

If F is a homotopical functor by the universal property of localization we obtain a unique functor such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & & \downarrow \\ \operatorname{Ho}\mathcal{C} & \underset{\exists !}{\dashrightarrow} & \operatorname{Ho}\mathcal{D}. \end{array}$$

[1] In the sense described in **Remark 5.7**.

Definition 5.10. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between homotopical categories $\mathcal{C}$ and $\mathcal{D}$.

1. A *total left derived functor* of F is a right Kan extension $LF := \mathrm{Ran}_\gamma \delta F$, where γ, δ are the localization functors corresponding to $\mathcal{C}, \mathcal{D}$ respectively. I.e

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ \mathrm{Ho}\mathcal{C} & \xrightarrow[LF=\mathrm{Ran}_\gamma \delta F]{} & \mathrm{Ho}\mathcal{D}. \end{array}$$

2. a *left derived functor* of F is a homotopical functor

$$\mathbb{L}F\colon \mathcal{C} \to \mathcal{D}$$

with a natural transformation $\lambda\colon \mathbb{L}F \to F$ such that $(\delta\mathbb{L}F, \delta\lambda)$ is a total left derived functor of F.

5.1 Derived functors using deformations.

In our discussion of model categories we describe cofibrant objects and cofibrant replacement. We will now generalize this notion to the setting of homotopical categories.

Definition 5.11. Let $\mathcal{C}$ be a homotopical category. A *left deformation* on $\mathcal{C}$ is a functor $Q\colon \mathcal{C} \to \mathcal{C}$ equipped with a natural weak equivalence $q\colon Q \to \mathrm{id}_\mathcal{C}$.

We advise the reader to think of a left deformation (Q, q) as a functorial cofibrant replacement in which for all $X \in \mathcal{C}$ the comparison maps q_X form a natural weak equivalence. Furthermore we define $\mathcal{M}_Q$ to be the homotopical category spanned by the objects in the image of Q. We advise the reader to think of $\mathcal{M}_Q$ as the category of cofibrant objects.

Remark 5.12. Any left deformation is homotopical by the 2-out-of-3 property of weak equivalences.

Definition 5.13. A *left deformation* for a functor $F\colon \mathcal{C} \to \mathcal{D}$ of homotopical categories is a left deformation on $\mathcal{C}$ such that F is homotopical on M_Q.

When we eventually discuss homotopy colimits one of the main results in proving homotopy invariance of our construction will be the following theorem.

Theorem 5.14. *If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor of homotopical categories and $q\colon Q \to \mathrm{id}_\mathcal{C}$ is a left deformation for F then $FQ = \mathbb{L}F$ is a left derived functor of F.*

Proof. Let $\delta\colon \mathcal{D} \to \mathrm{Ho}\mathcal{D}$ denote the localization of $\mathcal{D}$. We must show that $(\delta FQ, \delta Fq)$ satisfy the universal property of a right Kan extension in $\mathrm{Fun}(\mathrm{Ho}\mathcal{C}, \mathrm{Ho}\mathcal{D})$ which by **Remark 5.7** is equivalent to showing it is satisfied in $\mathrm{Fun}^{wti}(\mathcal{C}, \mathrm{Ho}\mathcal{D})$. Suppose $G\colon \mathcal{C} \to \mathrm{Ho}\mathcal{D}$ is homotopical and $\gamma\colon G \to \delta F$ is a natural transformation. Now Q is homotopical thus Qq is a natural isomorphism. From naturality of γ for all objects $c \in \mathcal{C}$ we get that

$$\begin{array}{ccc} G(c) & \xrightarrow{\gamma_c} & \delta Fc \\ {\scriptstyle Gq_c^{-1}}\downarrow & & \uparrow{\scriptstyle\delta Fq_c} \\ GQ(c) & \xrightarrow[\gamma Q_c]{} & \delta FQ(c) \end{array}$$

commute. These are all components of natural transformations hence γ factors through δFQ.

To see uniqueness, suppose that γ factors as

$$G \xrightarrow{\gamma'} \delta FQ \xrightarrow{\delta Fq} \delta F.$$

Then the restriction γ'_Q is uniquely determined. This follows since q_Q is a natural weak equivalence, then for objects in $\mathcal{C}$. Now since F is homotopical on $\mathcal{C}_Q$ it follows that δFq_Q is an isomorphism. Thus we conclude uniqueness of γ', since by naturality

$$\begin{array}{ccc} GQ(c) & \xrightarrow{\gamma'_{Qc}} & \delta FQ^2(c) \\ \downarrow{\scriptstyle Gq_c} & & \downarrow{\scriptstyle \delta FQq_c} \\ G(c) & \xrightarrow{\gamma'_c} & \delta FQ(c) \end{array}$$

commutes, furthermore the vertical maps are isomorphisms. □

Another technical result needed to prove homotopy invariance is the following lemma.

Lemma 5.15. *Let (Q, q) be a left deformation on $\mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{D}$ a functor of homotopical categories. If*

1. *FQ is homotopical*
2. *$FqQ\colon FQ^2 \to FQ$ is a natural weak equivalence*

then (Q, q) is a left deformation for F.

Proof. Let $f\colon c \to c'$ be a weak equivalence in M_Q by definition there exists $x, y \in M$ such that $Q(x) = c$ and $Q(y) = c'$. Now consider $Q(f)\colon Q^2(x) \to Q^2(y)$. This is again a weak equivalence, thus $FQ(f)$ is a weak equivalence by the first assumption. Now by the 2 out of 3 property and the second assumption we get that $F(f)$ is a weak equivalence. □

We would like homotopy colimits to be a good, homotopy invariant, approximation to the colimit functor. Therefore, these two results allow to form a strategy form defining homotopy colimits which should behave nicely with respect to homotopical properties of a category. The strategy will be that for every small category $\mathcal{D}$ to define a left deformation for the colimit functor

$$\mathcal{C}^{\mathcal{D}} \xrightarrow{\text{colim}} \mathcal{C}$$

6 The Bar Construction

During the section we discuss two sided simplicial bar construction and define the functor tensor product.

6.1 The functor tensor product

We will now define the functor tensor product.

Definition 6.1. Let $- \otimes -: \mathcal{V} \times \mathcal{M} \to \mathcal{M}$ be bifunctor and $F: \mathcal{D} \to \mathcal{M}$, $G: \mathcal{D}^{op} \to \mathcal{V}$ be functors. The *functor tensor product* of F and G is the coend

$$G \otimes_{\mathcal{D}} F := \int^{d \in \mathcal{D}} G(d) \otimes F(d)$$

Many of the results from the section on Kan extensions can be rephrased in terms of functor tensor products. We will not spend time on explaining this, but we will make an effort to note when we use the results in this form.

Remark 6.2. In the situation of **Definition 4.9** geometric realization can be expressed as the tensor product

$$\Delta^\bullet \otimes_{\mathbb{\Delta}^{op}} X_\bullet.$$

6.2 The bar construction

Construction 6.3 ([Rie14])**.** Suppose $\mathcal{M}$ is an simplicially enriched tensored category and $\mathcal{D}$ is a small category. If $G: \mathcal{D}^{op} \to s\mathcal{S}et$ and $F: \mathcal{D} \to \mathcal{M}$ are functors, then the *two sided simplicial bar construction* $B_\bullet(G, \mathcal{D}, F)$ is the simplicial object in $\mathcal{M}$ defined as

$$B_n(G, \mathcal{D}, F) = \coprod_{d:\ [n] \to \mathcal{D}} G(d(n)) \otimes F(d(0)).$$

Now for $\alpha: [n] \to [m]$ we want to produce a map

$$B_m(G, \mathcal{D}, F) \to B_n(G, \mathcal{D}, F).$$

Suppose we are given $d': [m] \to \mathcal{D}$ then $d'\alpha$ defines a functor $[n] \to \mathcal{D}$. This induces a map

$$G(d'(m)) \otimes F(d'(0)) \longrightarrow G(d'\alpha(n)) \otimes F(d'\alpha(0))$$

since there exists unique maps

$$d'(0) \to d'\alpha(0)$$
$$d'\alpha(n) \to d'(m).$$

Take this to be the map induced by α.

Definition 6.4 ([Rie14])**.** In the notation of **Construction 6.3** the *bar construction* is the geometric realization of the simplicial object defined in **Construction 6.3**

$$B(G, \mathcal{D}, F) := |B_\bullet(G, \mathcal{D}, F)|$$

The following result gives some intuition to why the two sided simplicial bar construction should be related to colimits.

Proposition 6.5. *The colimit*

$$\operatorname*{colim}_{\Delta^{op}} B_\bullet(G, \mathcal{D}, F)$$

is given by the functor tensor product $G \otimes_{\mathcal{D}} F$

Proof. The inclusion

$$\left([1] \underset{d_0}{\overset{d_1}{\rightrightarrows}} [0] \right) \longrightarrow \Delta^{op}$$

Is final thus by [Mac13, Thm. IX.3.1] the first of the following isomorphism hold

$$\begin{aligned}
\operatorname*{colim}_{\Delta^{op}} B_\bullet(G, \mathcal{D}, F) &\cong \operatorname{coeq}\left(\coprod_{f\colon a \to b \in \mathcal{D}} G(b) \otimes F(a) \rightrightarrows \coprod_{a \in \mathcal{D}} G(a) \otimes F(a) \right) \\
&\cong \int^{d \in \mathcal{D}} G(d) \otimes F(d) \\
&\cong G \otimes_{\mathcal{D}} F.
\end{aligned}$$

□

Remark 6.6. In particular we get that

$$\operatorname*{colim}_{\Delta^{op}} B_\bullet(*, \mathcal{D}, F) \cong * \otimes_{\mathcal{D}} F \cong \operatorname*{colim}_{d \in \mathcal{D}} F(d).$$

This allows us to think of $B_\bullet(*, \mathcal{D}, F)$ as a replacement of F by a simplicial object in $\mathcal{M}$. Which should be our first indication that the bar construction $B(*, \mathcal{D}, F)$ should be in someway related to $\operatorname{hocolim} F$. In fact if F is pointwise cofibrant it is indeed the case that $\operatorname{hocolim} F \cong B(*, \mathcal{D}, F)$.

Remark 6.7. The objects $B(\mathcal{D}(-, d), \mathcal{D}, F) \in \mathcal{M}$ depend functorially on $d \in \mathcal{D}$ we will denote the functor $\mathcal{D} \to \mathcal{M}$ which on objects is given by $d \mapsto B(\mathcal{D}(-, d), \mathcal{D}, F)$ by $B(\mathcal{D}, \mathcal{D}, F)$. Varying F this allows us to obtain an functor

$$B(\mathcal{D}, \mathcal{D}, -)\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{D}}$$

Lemma 6.8. *For* $G\colon \mathcal{D}^{op} \to s\mathcal{S}et$ *the following isomorphism hold:*

$$G \otimes_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, F) \cong B(G, \mathcal{D}, F)$$

Proof. This follows from the following string of isomorphisms:

$$\begin{aligned}
G \otimes_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, F) &= \int^{d \in \mathcal{D}} G(d) \otimes B(\mathcal{D}(-, d), \mathcal{D}, F) \\
&\cong \int^{d \in \mathcal{D}} G(d) \otimes \left(\int^{n \in \Delta} \Delta^n \otimes \coprod_{d\colon [n] \to \mathcal{D}} \mathcal{D}(d_n, d) \otimes F(d_0) \right)
\end{aligned}$$

using that $K \otimes -\colon M \to M$ is a left adjoint and thus commutes with coends, and applying Fubini's theorem for coends we get:

$$\begin{aligned}
&\cong \int^{n \in \Delta} \Delta^n \otimes \left(\int^{d \in \mathcal{D}} G(d) \otimes \left(\coprod_{d\colon [n] \to \mathcal{D}} \mathcal{D}(d_n, d) \otimes F(d_0) \right) \right) \\
&\cong \int^{n \in \Delta} \Delta^n \otimes \left(\int^{d \in \mathcal{D}} \coprod_{d\colon [n] \to \mathcal{D}} G(d) \otimes \mathcal{D}(d_n, d) \otimes F(d_0) \right).
\end{aligned}$$

Now since coends commutes with colimits we get that

$$
\begin{aligned}
&\cong \int^{n\in\mathbb{\Delta}} \Delta^n \otimes \left(\coprod_{d\colon [n]\to\mathcal{D}} \left(\int^{d\in\mathcal{D}} G(d) \times \mathcal{D}(d_n, d) \right) \otimes F(d_0) \right) \\
&\cong \int^{n\in\mathbb{\Delta}} \Delta^n \otimes \left(\coprod_{d\colon [n]\to\mathcal{D}} G(d_n) \otimes F(d_0) \right) \\
&= B(G, \mathcal{D}, F)
\end{aligned}
$$

where the last isomorphism is the coYoneda lemma. □

There is a similar statement about the 3rd variable of the bar construction, replacing contravariant representable functors with covariant ones.

7 Homotopy Colimits

In this section we will give a construction of the left derived functor of colim: $\mathcal{M}^{\mathcal{D}} \to \mathcal{M}$, for some small category $\mathcal{D}$, in the case where $\mathcal{M}$ is a simplicial model category. Furthermore we give a model for the homotopy colimit of a functor $F\colon \mathcal{D} \to \mathcal{M}$. For the entire section let $\mathcal{M}$ be a simplicial model category, equipped with a cofibrant replacement $Q\colon \mathcal{M} \to \mathcal{M}$ and $\mathcal{D}$ a small category. In particular we can consider $\mathcal{M}^{\mathcal{D}}$ a homotopical category with levelwise weak equivalences as weak equivalences.

Lemma 7.1. *If $F\colon \mathcal{D} \to \mathcal{M}$ is pointwise cofibrant then the two sided simplicial bar $B_\bullet(*, \mathcal{D}, F)$ is Reedy cofibrant.*

Proof. We show that for all n the canonical map $L_n B_\bullet(*, \mathcal{D}, F) \to B_n(*, \mathcal{D}, F)$ is a cofibration. The n-th latching object is given by the colimit

$$\operatorname*{colim}_{[n]\to[k]} B_k(*, \mathcal{D}, F).$$

For $n \geq 2$ the indexing category has a cofinal subcategory given by the full subcategory restricted to $k = n-1$ and $k = n-2$ (this is shown in [Hir09, Prop. 15.2.6]) hence

$$L_n B_\bullet(*, \mathcal{D}, F) \cong \operatorname*{colim}_{\substack{[n]\to[n-1]\\ [n]\to[n-2]}} B_\bullet(*, \mathcal{D}, F)|_{n-1,n-2}$$

This is exactly

$$\coprod_{d\in(N(\mathcal{D})^d)_n} F(d_0)$$

where $(N(\mathcal{D})^d)_n$ is the degenerate n-simplicies in $N(\mathcal{D})$. Thus since F is pointwise cofibrant and cofibrations are closed under coproducts and pushouts the map

$$L_n B_\bullet(*, \mathcal{D}, F) \to B_n(*, \mathcal{D}, F)$$

is a cofibration. □

Remark 7.2. Since $|-|\colon s\mathcal{M} \to \mathcal{M}$ is left Quillen (see **Corrolary 4.23**) it follows that for any natural weak equivalence $F \xrightarrow{\sim} F'$, with F, F' pointwise cofibrant that $B(*, \mathcal{D}, F) \xrightarrow{\sim} B(*, \mathcal{D}, F')$ is a weak equivalence.

Remark 7.3. Since tensoring with a simplicial sets preserves cofibrant objects it follows that for any functor $G\colon \mathcal{D}^{op} \to s\mathcal{S}et$ and $F\colon \mathcal{D} \to \mathcal{M}$ pointwise cofibrant then $B_\bullet(G, \mathcal{D}, F)$ is Reedy cofibrant.

Theorem 7.4. *The functor $B(G, \mathcal{D}, -)$ is homotopical on the full subcategory of $\mathcal{M}^{\mathcal{D}}$ spanned by pointwise cofibrant functors. In particular $B(G, \mathcal{D}, Q-)$ is homotopical.*

Proof. This follows from **Lemma 7.1**, **Remark 7.2** and **Remark 7.3**. □

The goal is now to prove that $B(\mathcal{D}, \mathcal{D}, Q-)$ is a left deformation on $\mathcal{M}^{\mathcal{D}}$ and that the natural weak equivalence

$$B\left(\mathcal{D}, \mathcal{D}, QB(\mathcal{D}, \mathcal{D}, Q-)\right) \to B(\mathcal{D}, \mathcal{D}, Q-)$$

is preserved by $\operatorname{colim}_{\mathcal{D}}\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}$. Since by **Lemma 5.15** this would imply that this is in fact a left deformation for $\operatorname{colim}_{\mathcal{D}}$.

Proposition 7.5. *The simplicial object $B_\bullet(\mathcal{D}(-, d), \mathcal{D}, F)$ admits an augmentation and extra degeneracy such that the augmentation is a retract of the extra degeneracy.*

Proof. Let $B_{-1}(\mathcal{D}(-,d),\mathcal{D},F) \coloneqq F(d)$. For $d' \in \mathcal{D}$ consider the map

$$\mathcal{D}(d',d) \to \mathcal{M}(F(d'),F(d))$$

given by $f \mapsto F(f)$. The transpose $\bar{f}\colon \mathcal{D}(d',d)\otimes F(d') \to F(d)$ defines a map

$$B_\bullet(\mathcal{D}(-,d),\mathcal{D},F) \xrightarrow{r} F(d)$$

where we consider $F(d)$ a constant simplicial object in $\mathcal{M}$. Now to define the extra degeneracy for $n \geq 0$ consider the map $B_n(\mathcal{D}(-,d),\mathcal{D},F) \to B_{n+1}(\mathcal{D}(-,d),\mathcal{D},F)$ induced by inserting id_d at the beginning of any n-simplex $d\colon [n] \to \mathcal{D}$ with $d_0 = d$. For $n = -1$ consider the map

$$F(d) \cong * \otimes F(d) \xrightarrow{(\mathrm{id}_d)^* \otimes F(d)} \mathcal{D}(d,d)\otimes F(d) \xrightarrow{i_d} \coprod_{d'} \mathcal{D}(d',d)\otimes F(d') = B_1(\mathcal{D}(-,d),\mathcal{D},F)$$

It is clear from how we have defined things that

$$B_{-1}\left(\mathcal{D}(-,d),\mathcal{D},F\right) \xrightarrow{s_0} B_\bullet(\mathcal{D}(-,d),\mathcal{D},F) \xrightarrow{r} B_{-1}(\mathcal{D}(-,d),\mathcal{D},F)$$

is the identity. □

Remark 7.6. Note that r is natural in d, furthermore s_0 need not be natural in $\mathcal{D}$. Hence we get a natural transformation

$$\varepsilon\colon B(\mathcal{D},\mathcal{D},F) \to F.$$

Proposition 7.7. *The natural transformation ε is a natural weak equivalence.*

Proof. This is an application of [Rie14, Corollary 4.5.4] to the augmentation and extra degeneracy defined in **Proposition 7.5**. □

Remark 7.8. For the last part of this section we will let ε_F be the natural weak equivalence $B(\mathcal{D},\mathcal{D},F) \xrightarrow{\sim} F$. By naturality we get that

$$\begin{array}{ccc} B(\mathcal{D},\mathcal{D},QB(\mathcal{D},\mathcal{D},QF)) & \xrightarrow{\varepsilon_{QB}} & QB(\mathcal{D},\mathcal{D},QF) \\ {\scriptstyle B(\mathcal{D},\mathcal{D},q)}\downarrow & & \downarrow{\scriptstyle q} \\ B(\mathcal{D},\mathcal{D},B(\mathcal{D},\mathcal{D},QF)) & \xrightarrow{\varepsilon_B} & B(\mathcal{D},\mathcal{D},QF) \end{array}$$

commutes. Hence by the 2-of-3 property $\operatorname{colim}_{\mathcal{D}} \varepsilon_{QB}$ is an weak equivalence if and only if $\operatorname{colim}_{\mathcal{D}} \varepsilon_B$ is.

Proposition 7.9. *If F is pointwise cofibrant, then* $\operatorname{colim}_{\mathcal{D}}$ *preserves the weak equivalence*

$$\varepsilon_B\colon B(\mathcal{D},\mathcal{D},B(\mathcal{D},\mathcal{D},F)) \to B(\mathcal{D},\mathcal{D},F).$$

Proof. By naturality we see that the map ε_B is given by the composition

$$B(\mathcal{D},\mathcal{D},F) \to B(\mathcal{D},\mathcal{D},\mathcal{D}) \otimes_{\mathcal{D}} F \xrightarrow{\varepsilon\otimes F} \mathcal{D}\otimes F \to F$$

where we apply that $\mathcal{D}\otimes_{\mathcal{D}} F \cong F$ by the coYoneda lemma and ε is the weak equivalence obtained by the extra degeneracy and augmentation defined on

$$B_\bullet(\mathcal{D}(-,d'),\mathcal{D},\mathcal{D}(d,-))$$

into $\mathcal{D}(d,d')$. Thus by applying naturality and **Lemma 6.8** we get that

$$\begin{array}{ccc}
* \otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},B(\mathcal{D},\mathcal{D},F)) & \xrightarrow{*\otimes_{\mathcal{D}}\varepsilon_B} & * \otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},F) \\
\downarrow & & \downarrow \\
* \otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},\mathcal{D}) \otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},F) & \xrightarrow{*\otimes_{\mathcal{D}}\varepsilon\otimes_{\mathcal{D}}B} & * \otimes_{\mathcal{D}} \mathcal{D} \otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},F) \\
\downarrow & & \downarrow \\
B(*,\mathcal{D},\mathcal{D}) \otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},F) & \xrightarrow{\varepsilon\otimes_{\mathcal{D}}B} & * \otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},F) \\
\downarrow & & \downarrow \\
B(B(*,\mathcal{D},\mathcal{D}),\mathcal{D},F) & \xrightarrow{B(\varepsilon,\mathcal{D},F)} & B(*,\mathcal{D},F)
\end{array}$$

commutes. Now the vertical maps are all isomorphisms hence weak equivalence, thus by 2-of-3 it is sufficient for $B(\varepsilon,\mathcal{D},F)$ to be a weak equivalence to prove the statement. This is follows directly from ε being a weak equivalence and **Theorem 7.4**. □

Putting together the work of this chapter we get the following theorem.

Theorem 7.10 (Theorem 5.1.1 [Rie14]). *Let $\mathcal{M}$ be a simplicial model category with cofibrant replacement $Q\colon \mathcal{M} \to \mathcal{M}$. Then the pair*

$$B(\mathcal{D},\mathcal{D},Q-)\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{D}} \qquad B(\mathcal{D},\mathcal{D},Q-) \xrightarrow{\varepsilon_Q} Q(-) \xrightarrow{q} \mathrm{id}_{\mathcal{M}^{\mathcal{D}}}$$

is a left deformation for $\operatorname{colim}_{\mathcal{D}}\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}$. *In particular* $\operatorname{colim}_{\mathcal{D}}$ *admits a left derived functor* $\mathbb{L}\operatorname{colim}_{\mathcal{D}}$.

Proof. By **Proposition 7.9** and **Theorem 7.4** we see the pair satisfies the conditions of **Lemma 5.15** hence is a left deformation for $\operatorname{colim}_{\mathcal{D}}$. In particular by **Theorem 5.14** it follows that

$$\mathbb{L}\operatorname*{colim}_{\mathcal{D}} = \operatorname*{colim}_{\mathcal{D}} B(\mathcal{D},\mathcal{D},Q-)$$

is a left derived functor of $\operatorname{colim}_{\mathcal{D}}\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}$. □

Definition 7.11. If $\mathcal{M}$ is a simplicial model category and $\mathcal{D}$ is a small category, then we define the homotopy colimit

$$\operatorname*{hocolim}_{\mathcal{D}} := \mathbb{L}\operatorname*{colim}_{\mathcal{D}}$$

as the left derived functor of the colimit functor.

We have now abstractly defined the $\operatorname{hocolim}_{\mathcal{D}}$ and shows that it satisfies the universal property one would expect of it. We will finish end this section with giving a formula for computing homotopy colimits.

Proposition 7.12. *The homotopy colimit is given by*

$$\operatorname*{hocolim}_{\mathcal{D}} \cong B(*,\mathcal{D},Q-)$$

Proof. By **Theorem 7.10** the homotopy colimit is given by the functor $\operatorname{colim}_{\mathcal{D}} B(\mathcal{D},\mathcal{D},Q-)$. By applying **Lemma 6.8** it follows from straight computation that

$$\begin{aligned}
\operatorname*{colim}_{\mathcal{D}} B(\mathcal{D},\mathcal{D},Q-) &\cong * \otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},Q-) \\
&\cong B(* \otimes_{\mathcal{D}} \mathcal{D},\mathcal{D},Q-) \\
&\cong B(*,\mathcal{D},Q-)
\end{aligned}$$

□

Remark 7.13. In this thesis we have only defined homotopy colimits, but there is of course a dual notion of a homotopy limit. This can be defined as a right derived functor of the functor $\lim_{\mathcal{D}}\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}$, which can be constructed in a similar way.

A Ends and Coends

In this Appendix we will recall basic results on Ends and Coends starting with the definition

Definition A.1. Let $H\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor. A *cowedge* consists of an object $X \in \mathcal{D}$ along with maps $\omega_c\colon H(c,c) \to X$ for all $c \in \mathcal{C}$ such that

$$\begin{array}{ccc} H(c',c) & \xrightarrow{f_*} & H(c',c') \\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow \\ H(c,c) & \longrightarrow & X \end{array}$$

commutes. The *coend* of H denoted

$$\int^{c\in\mathcal{C}} H(c,c)$$

if it exists, ss an object of $\mathcal{D}$ with maps $H(c,c) \to \int^{c\in\mathcal{C}} H(c,c)$ for all $c \in \mathcal{C}$ such that for all $f\colon c \to c'$ in $\mathcal{C}$

$$\begin{array}{ccc} H(c',c) & \xrightarrow{f_*} & H(c',c') \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow \\ H(c,c) & \longrightarrow & \int^{c\in\mathcal{C}} H(c,c) \end{array}$$

commutes universally. I.e. such that $\int^{c\in\mathcal{C}} H(c'c)$ is the initial cowedge on H in the evident category of cowedges on H.

Remark A.2. Here we have adopted the notation usually used when considering representable functors.

Remark A.3. Additionally there is dual notion of an end. Which we will denote

$$\int_{c\in\mathcal{C}} H(c,c)$$

Proposition A.4. *C If $F, G\colon \mathcal{C} \to \mathcal{D}$ are functors then we have a natural isomorphism*

$$\int_{c\in\mathcal{C}} \mathcal{D}(F(-), G(-)) \cong \mathrm{Nat}(F, G)$$

Proof. Suppose we have a wedge $\omega\colon X \dot{\to} \mathcal{D}(F(-), G(-))$ then for $f\colon c \to c'$ we have a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\omega_c} & \mathcal{D}(F(c), G(c)) \\ {\scriptstyle \omega_{c'}}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ \mathcal{D}(F(c'), G(c')) & \xrightarrow[f^*]{} & \mathcal{D}(F(c), G(c')). \end{array}$$

Hence for any $x \in X$ we get that $G(f)\omega_c(x) = \omega_{c'}(x)F(f)$ i.e a naturality square

$$\begin{array}{ccc} F(c) & \xrightarrow{\omega_c(x)} & G(c) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ F(c') & \xrightarrow[\omega_{c'}(x)]{} & G(c'). \end{array}$$

This defines a function $X \to \mathrm{Nat}(F, G)$ assigning the natural transformation $\omega(x)$, with components $\omega_c(x)$, for each $x \in X$. It is clear that $\mathrm{Nat}(F, G)$ can be a used to define a wedge on $\mathcal{D}(F(-), G(-))$. Checking that $\mathrm{Nat}(F, G)$ is terminal among such wedges is just a matter of checking uniqueness of $X \to \mathrm{Nat}(F, G)$. $\square$

To state some basic results on coends we will first prove the following characterization.

Proposition A.5. *Let $\mathcal{D}$ be a category with coproducts and coequalizers, $\mathcal{C}$ a small category and $H\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor, then*

$$\int^{c\in\mathcal{C}} H(c, c) \cong \mathrm{coeq}\left(\coprod_{f\colon c\to c'} H(c', c) \rightrightarrows \coprod_{c\in\mathcal{C}} H(c, c) \right)$$

with the parallel morphisms induced by the maps $i_c f^*$ and $i_{c'} f_*$, where i denotes the injections.

Proof. We will proceed by showing that these satisfy the same universal property. There is a canonical map

$$\coprod_{c\in\mathcal{C}} H(c, c) \to \int^{c\in\mathcal{C}} H(c, c).$$

Now since the following diagram is commutative

$$\begin{array}{ccccc}
H(c', c) & \xrightarrow{f^*} & H(c', c') & \searrow & \\
\downarrow & & \downarrow & & \\
\coprod_{f\colon c\to c'} H(c', c) & \rightrightarrows & \coprod_{c\in\mathcal{C}} H(c, c) & \longrightarrow & \int^{c\in\mathcal{C}} H(c, c) \\
\uparrow & & \uparrow & & \\
H(c', c) & \xrightarrow[f_*]{} & H(c, c) & \nearrow &
\end{array}$$

we get a induced map

$$\mathrm{coeq}\left(\coprod_{f\colon c\to c'} H(c', c) \rightrightarrows \coprod_{c\in\mathcal{C}} H(c, c) \right) \longrightarrow \int^{c\in\mathcal{C}} H(c, c).$$

We will now produce a map in the opposite direction. From which showing that this pair gives an isomorphism is standard. We will denote the composite

$$H(c, c) \to \coprod_{c\in\mathcal{C}} H(c, c) \to \mathrm{coeq}\left(\coprod_{f\colon c\to c'} H(c', c) \rightrightarrows \coprod_{c\in\mathcal{C}} H(c, c) \right)$$

by k_c. This defines a cowedge, since the last map coequalizes the two maps. Hence we get the intended map. Finishing the proof. $\square$

Remark A.6. There is a dual result for ends.

This result now gives us the following results

Theorem A.7. *We have*

1. *(Co)continuous functors preserve (co)ends.*

2. Hom *functors take ends to ends in the covariant variable and coends to ends in the contravariant variable.*

3. *There exists a Fubini's theorem for (co)ends i.e if we are given a functor* $H\colon \mathcal{D}^{op} \times \mathcal{D} \times \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{P}$ *then the following isomorphisms hold*

$$\int^{(d,c)\in\mathcal{D}\times\mathcal{C}} H(d,d,c,c) \cong \int^{d\in\mathcal{D}} \int^{c\in\mathcal{C}} H(d,d,c,c) \cong \int^{c\in\mathcal{C}} \int^{d\in\mathcal{D}} H(d,d,c,c)$$

and similarly for ends.

We omit the proof, but this can all be found in [Mac13] or [Lor15].

References

[Qui06] Daniel G Quillen. *Homotopical algebra*. Vol. 43. Springer, 2006.

[GJ09] Paul G Goerss and John F Jardine. *Simplicial homotopy theory*. Springer Science & Business Media, 2009.

[Hir09] Philip S Hirschhorn. *Model categories and their localizations*. 99. American Mathematical Soc., 2009.

[Mac13] Saunders Mac Lane. *Categories for the working mathematician*. Vol. 5. Springer Science & Business Media, 2013.

[Rie14] Emily Riehl. *Categorical homotopy theory*. 24. Cambridge University Press, 2014.

[Lor15] Fosco Loregian. “This is the (co) end, my only (co) friend”. In: *arXiv preprint arXiv:1501.02503* (2015).

[Nie19] Marius Nielsen. “Tannaka Duality for Symmetric Fusion Categories”. Bachelors level project out of course scope. Supervised by Thomas Wasserman. 2019. URL: https://sites.google.com/view/mariusnielsen/start.