CHROMATIC HOMOTOPY THEORY, FROM THE PERSPECTIVE OF FORMAL GROUPS

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Abstract. In this talk we will give a modern overview of chromatic homotopy theory and its relation to formal groups. Throughout this talk we will introduce the moduli stack of formal groups $\mathcal{M}_{fg}$, discus the height filtration on $\mathcal{M}_{fg}$, then we will go on to discuss the thick subcategory theorem.

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This note is written as part of the preparation for my exam in my seminar course at NTNU on Chromatic homotopy theory.

1. The Adams spectral sequence

Getting to information on the stable homotopy category has historically been a very complicated question. Even the stable homotopy groups of spheres $\pi_*\mathbb{S}$ are largely unknown. One way to attack this question is through the MU-based Adams spectral sequence, which has signature

$$E_1^{s,t} = \pi_s(MU^{\otimes t})$$

and converges to $\pi_{s-t}(\mathbb{S})$. Now the E_1-page of this spectral sequence is very complicated. However, it turns out that one E_2-page is given by

$$E_2^{s,2t} = H^s(\mathcal{M}_{fg}, \omega^t).$$

Here $\mathcal{M}_{fg}$ is the moduli stack of formal groups and ω is a certain line bundle on $\mathcal{M}_{fg}$. It is reasonable to believe that one might learn about the stable homotopy groups of spheres from studying the geometry of the moduli stack of formal groups. The goal for this note will be to introduce formal groups and define their moduli stack. Then we will introduce the height filtration on $\mathcal{M}_{fg}$. Finally, we discuss how the height filtration, can be used to state the thick subcategory theorem.

Conventions. In this note we will denote to the ∞-category of spaces by $\mathcal{S}$, the ∞-category of spectra by $\mathcal{S}p$. We will denote the 1-category of discrete commutative rings by CAlg and the ∞-category of commutative ring spectra ($\mathbb{E}_\infty$-algebras in $\mathcal{S}p$ to be precise) by $\mathrm{CAlg}(\mathcal{S}p)$. We will always consider $\mathrm{Aff} = \mathrm{CAlg}^{\mathrm{op}}$ a site with the étale topology, and we denote the Yoneda embedding

$$\mathrm{Aff} \to \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$$

by $\mathrm{Spec}(-)$.

2. Stacks and formal groups

Definition 2.1. A *stack* is a presheaf $X\colon \mathrm{CAlg} \to \mathcal{S}$ which is a sheaf with respect to the étale topology.

Example 2.2. The fpqc-topology on Aff is subcanonical, which implies that the etale topology is subcanonical. It follows that the Yoneda embedding $\mathrm{Spec}(-)$, takes values in the full subcategory Stk of $\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$, spanned by etale sheaves. Now since open immersions are etale it follows that the category of schemes Sch embeds fully faithfully into Stk.

Remark 2.3. Note that the above definition is simply a generalization of the classical definition. This follows since ∞-category of spaces comes with a filtration of subcategories

$$\mathcal{S}_{\leqslant 0} \to \mathcal{S}_{\leqslant 1} \to \cdots \to \mathcal{S}$$

where $\mathcal{S}_{\leqslant n}$ denotes the full subcategory spanned by n-truncated spaces. All of these functors are right adjoints. Which implies that for any site $\mathcal{C}$ the ∞-category

$$\mathrm{Shv}_{\mathcal{S}_{\leqslant n}}(\mathcal{C})$$

embeds into

$$\mathrm{Shv}_{\mathcal{S}_{\leqslant k}}(\mathcal{C})$$

for any $k \geqslant n$. Furthermore, the same holds when we replace $\mathcal{S}_{\leqslant k}$ by $\mathcal{S}$. Now $\mathcal{S}_{\leqslant 0} \simeq \mathrm{Set}$ and $\mathcal{S}_{\leqslant 1} \simeq \mathrm{Grpd}$.

Definition 2.4. For a stack X, we denote the ∞-category of stacks over X by $\mathrm{Stk}_{/X}$.

Remark 2.5. For a ring R, then $\mathrm{Stk}_{/\mathrm{Spec}(R)} \simeq \mathrm{Shv}(\mathrm{CAlg}_R^{\mathrm{op}})$ with respect to the etale topology on $\mathrm{CAlg}_R^{\mathrm{op}}$.

Definition 2.6. For a ring R the sheaf of *nilpotents* $\mathrm{Nil}_R\colon \mathrm{CAlg}_R \to \mathrm{Set}$ is the sheaf which takes an R-algebra S to the set $\mathrm{Nil}(S)$ of nilpotent elements of S.

Definition 2.7. For a ring R a *formal group law* on R is a lift

$$\begin{tikzcd} & \mathrm{Ab} \arrow[d, "U"] \\ \mathrm{CAlg}_R \arrow[r, "\mathrm{Nil}_R"'] \arrow[ur, dotted, "F"] & \mathrm{Set}. \end{tikzcd}$$

along the forgetful functor $U\colon \mathrm{Ab} \to \mathrm{Set}$.

Remark 2.8. Note that since the forgetful functor $U\colon \mathrm{Ab} \to \mathrm{Set}$ creates limits, it follows that any formal group law F over R is an etale sheaf on $\mathrm{CAlg}_R^{\mathrm{op}}$.

Definition 2.9. Let R be a ring. A *formal group* over R is an etale sheaf $G\colon \mathrm{CAlg}_R \to \mathrm{Ab}$, such that there exists a Zariski covering

$$\coprod_{i=1}^{n} \mathrm{Spec}(R[f_i^{-1}]) \to \mathrm{Spec}(R)$$

with the property that the composite

$$\mathrm{CAlg}_{R[f_i^{-1}]} \to \mathrm{CAlg}_R \xrightarrow{G} \mathrm{Ab}$$

is naturally isomorphic to a formal group law for every i.

Example 2.10. If $E \in \mathrm{CAlg}(\mathcal{S}p)$ is an even spectrum, then the formal scheme

$$\mathrm{Spf}(E^*(\mathbb{C}P^\infty))$$

admits the structure of a formal group, coming from the multiplication from the tensor product of complex line bundles $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$.

For the rest of this section we will fix a prime p and assume that R is an $\mathbb{F}_p$-algebra. Now, recall that for any $\mathbb{F}_p$-algebra R, the Frobenius map

$$\mathrm{Frob}_R\colon R \to R$$

given by $r \mapsto r^p$, is a ring homomorphism. Furthermore, for any map $f\colon R \to S$ of $\mathbb{F}_p$-algebras we have that

$$f \circ \mathrm{Frob}_R = \mathrm{Frob}_S \circ f.$$

So the Frobenius map defines a natural transformation, from the identify functor on CAlg_R to it self.

Definition 2.11. Let $X \in \mathrm{Stk}_{/\mathrm{Spec}(R)}$ be an étale sheaf over R, then the *Frobenius endomorphism* on X is the wiskered natural transformation

$$\begin{tikzcd} \mathrm{CAlg}_R \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow \mathrm{Frob}_R"] & \mathrm{CAlg}_R \arrow[r, "X"] & \mathcal{S}. \end{tikzcd}$$

Remark 2.12. From the above argument it follows that the Frobenius map induces a natural transformation from the identity on $\mathrm{Stk}_{/\mathrm{Spec}(R)}$ to itself.

Definition 2.13. Let X be an etale sheaf over Y, both of which are sheaves over an $\mathbb{F}_p$-algebra R. Then we define the *relative Frobenius* $\mathrm{Frob}_{X/Y}\colon X \to \mathrm{Frob}_Y^* X$ to be the map defined by the following diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{Frob}_{X/Y}"] \arrow[rr, bend left, "\mathrm{Frob}_X"] \arrow[dr] & \mathrm{Frob}_Y^* X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X \arrow[d] \\ & Y \arrow[r, "\mathrm{Frob}_Y"'] & Y. \end{tikzcd}$$

Definition 2.14. Let X be an etale sheaf over Y, both of which are sheaves over $\mathrm{Spec}(R)$. Then we define the *n'th relative Frobenius* to be

$$\mathrm{Frob}^n_{X/Y}\colon X \to (\mathrm{Frob}^n_Y)^* X$$

obtained via the above procedure applied to the n-fold composite of the Frobenius with it self.

We are now able to define *height*, this will later give allow us to define the height filtration on $\mathcal{M}_{fg}$ which in turn allows for a definition of the chromatic filtration on $\mathcal{S}p$.

Definition 2.15. Let R be an $\mathbb{F}_p$-algebra and G be a formal group over R. We say that G is *of height at least* n, if the multiplication by p map $[p]\colon G \to G$ factors through the n relative Frobenius. That is, there is a commuting diagram

$$\begin{tikzcd} G \arrow[r, "\mathrm{Frob}^n_{G/R}"] \arrow[dr, "p"'] & (\mathrm{Frob}^n_R)^* G \arrow[d, dotted] \\ & G. \end{tikzcd}$$

Remark 2.16. Classically height is defined locally in terms of coefficients the p'th power series $[p]_F(x)$ and this can be recovered from from our definition of height. However, this definition has the benefit that it is coordinate free.

Definition 2.17. Let G be a formal group over an $\mathbb{F}_p$-algebra R, then we say that G is *of height exact* n if G is of height at least n and the map

$$(\mathrm{Frob}^n_R)^* G \to G$$

is an isomorphism of formal groups. We say that G is *of infinite height* if it of height at least n for every $n \geqslant 0$.

3. The moduli stack of formal groups

We will now define the moduli stack of formal groups $\mathcal{M}_{fg}$ and the height filtration of formal groups. Finally, we will discuss how this can be used to classify the thick subcategories of $\mathcal{S}p^\omega$.

Definition 3.1. The *moduli stack of formal groups* $\mathcal{M}_{fg}\colon \mathrm{CAlg} \to \mathcal{S}$ is the etale sheaf of spaces which to any ring R associates the groupoid of formal groups over $\mathrm{Spec}(R)$ and their isomorphisms.

We will want to argue that $\mathcal{M}_{fg}$ is algebraic (that is informally not to far from being a scheme). In order to do so we will introduce an analogue of $\mathcal{M}_{fg}$ for formal group laws, and argue that this gives us as presentation of $\mathcal{M}_{fg}$.

Definition 3.2. The *moduli stack of formal group laws* $\mathcal{F}GL$ is the etale sheaf, which to a ring R associates the groupoid of formal group laws and their isomorphisms.

Remark 3.3. The moduli stack of formal group laws is actually representable by a ring L. This ring is called the *Lazard ring.*

Remark 3.4. If F is a formal group law, then considering F as a formal group defines a map

$$\mathcal{F}GL \to \mathcal{M}_{fg}.$$

This map is an effective epimorphism (which can be seen by seeing that it is a surjection of sheaves of path components). In particular, it follows that the augmented Cech nerve

$$\ldots \mathcal{F}GL \times_{\mathcal{M}_{fg}} \mathcal{F}GL \times_{\mathcal{M}_{fg}} \mathcal{F}GL \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{F}GL \times_{\mathcal{M}_{fg}} \mathcal{F}GL \rightrightarrows \mathcal{F}GL \longrightarrow \mathcal{M}_{fg}$$

is a colimit diagram in Stk.

We will now study this map in order to show that $\mathcal{M}_{fg}$ is algebraic.

Proposition 3.5. *The map $\mathcal{FGL} \to \mathcal{M}_{fg}$ is faithfully flat, affine and*

$$\mathcal{FGL} \times_{\mathcal{M}_{fg}} \mathcal{FGL} \simeq \mathcal{FGL} \times \mathbb{G}_{inv}$$

where $\mathbb{G}_{inv}$ is the sheaf of invertible powerseries.

Remark 3.6. Since $\mathcal{FGL} \simeq \mathrm{Spec}(L)$ there is an induced action by $\mathbb{G}_{inv}$ on $\mathcal{FGL}$.

Corollary 3.7. *The moduli stack of formal groups $\mathcal{M}_{fg}$ is the quotient of $\mathcal{FGL}$ by the action of $\mathbb{G}_{inv}$.*

The following theorem, due to Quillen, now explains how we get at the stable homotopy category.

Theorem 3.8 (Quillen)**.** *The Lazard ring L is isomorphic to MU_*.*

Using Quillens theorem and that $\mathrm{QCoh}\colon \mathrm{Stk}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ has good descent properties we now see that:

Theorem 3.9. *There is a symmetric monoidal equivalence of abelian categories*

$$\mathrm{Comod}^{even}_{MU_*MU} \xrightarrow{\sim} \mathrm{QCoh}(\mathcal{M}_{fg}).$$

Remark 3.10. The proof here really relies on a little bit more that what is described in this note. The problem is that we really to take grading into account.

Remark 3.11. This shows that there is a functor

$$\mathcal{F}_{(-)}\colon \mathrm{Sp} \to \mathrm{QCoh}(\mathcal{M}_{fg})$$

given by taking MU homology and composing with the above equivalence. From this one deduces that there are isomorphisms

$$\mathrm{Ext}^{s,t}(MU_*, MU_*MU) \cong \mathrm{Ext}^s(\mathcal{O}_{\mathcal{M}_{fg}}, \omega^t) \cong H^s(\mathcal{M}_{fg}, \omega^t)$$

for some line bundle ω. This is a somewhat explicit line bundle, which we will not spend time on in this note.

Now on to the height filtration, which allows us to filter $\mathcal{S}p$ using the functor $\mathcal{F}_{(-)}$.

Definition 3.12. The *height filtration* on $\mathcal{M}_{fg}$ is the filtration on $\mathcal{M}_{fg}$ where the n'th piece is the substack $\mathcal{M}^{\geqslant n}_{fg}$ spanned by formal groups of height at least n.

Proposition 3.13. *The map*

$$\mathcal{M}^{\geqslant n}_{fg} \to \mathcal{M}_{fg}$$

is an affine closed inclusion.

Remark 3.14. Since there is a different definition of height for every prime p. It is beneficial to study the induced height filtration on

$$\mathcal{M}_{fg} \times \mathrm{Spec}(\mathbb{Z}_{(p)})$$

instead.

4. Relations to stable homotopy theory

We now look towards stable homotopy theory and state a formulation of the famous subcategory theorem.

Definition 4.1. Let $\mathcal{M}^{\leqslant n}_{fg}$ be the stack theoretical compliment of $\mathcal{M}^{\geqslant n}_{fg}$ in $\mathcal{M}_{fg}$.

Definition 4.2. We say that a spectrum $X \in \mathcal{S}p$ is of height at most n if $\mathcal{F}_X$ is supported on $\mathcal{M}^{\leqslant n}_{fg}$. We denote the full subcategory a p-local spectra of height at most n by $\mathcal{S}p_n$.

Theorem 4.3 (Thick subcategory theorem)**.** *Let $\mathcal{C}$ be a thick subcategory of $\mathcal{S}p^{\omega}_{(p)}$. In this situation, there exists an integer $n \geqslant 0$ such that $C = \mathcal{S}p_n$.*