IDEMPOTENT ALGEBRAS AND STABLE RECOLLEMENT

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These are my notes for my talk 'Idempotent Algebras and Stable Recolloement' in Topics in Algebraic Topology, at UCPH 2021/2022.

For the entirety of this note, $\mathrm{map}(-,-)$ will denote the mapping spectrum, which is a lift along $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{An}$ of the mapping space $\mathrm{Map}(-,-)$.

1. Idempotent Algebras

For this section we fix a symmectric monoidal ∞-category $\mathcal{C}^\otimes$ and a $\mathbb{E}_0$-algebra $\eta\colon 1 \to E$.

Definition 1.1. The $\mathbb{E}_0$-algebra $\eta\colon 1 \to E$ is said to to be *idempotent* if the map

$$E \simeq 1 \otimes E \xrightarrow{\eta \otimes E} E \otimes E$$

is an equivalence.

Proposition 1.2. *The following are equivalent*

(1) The $\mathbb{E}_0$-algebra $\eta\colon 1 \to E$ is idempotent.
(2) The functor $E \otimes -\colon \mathcal{C} \to \mathcal{C}$ is a localization.

In this situation, we may simultaneously promote the essential image $L\mathcal{C}$ of $E \otimes -\colon \mathcal{C} \to \mathcal{C}$ and the functor $L\colon \mathcal{C} \to L\mathcal{C}$ to a symmetric monoidal ∞-category and a symmectric monoidal functor.

Proof. By [**lurie2009higher**] it suffices to show that the natural transformations

$$E \otimes - \xrightarrow{\eta \otimes -} E \otimes E \otimes -$$

$$E \otimes - \xrightarrow{E \otimes \eta} E \otimes E \otimes -$$

are natural equivalences, in order to show that 1) implies 2). But this is by assumption as $\mathcal{C}^\otimes$ is symmetric monoidal. Conversely, we see that $\eta\colon 1 \to E$ is an idempotent.

To prove the statement concerning symmetric monoidal structures it follows from [**lurie2017higher**], that it suffices to show that if a map $f\colon c \to c'$ is mapped to an equivalence by L, then so is $f \otimes d\colon c \otimes d \to c' \otimes d$ for any $d \in \mathcal{C}$. This is obvious, as $L(c) = E \otimes c$. $\square$

By the above proposition we have an adjunction

$$\mathcal{C}^\otimes \underset{i^\otimes}{\overset{L^\otimes}{\rightleftarrows}} L\mathcal{C}^\otimes$$

with $L^\otimes$ symmetric monoidal, and hence, $i^\otimes$ lax symmetric monoidal. In particular, we get a fully faithful functor

$$\mathrm{CAlg}(L\mathcal{C}) \hookrightarrow \mathrm{CAlg}(\mathcal{C}).$$

In particular, we can promote E to a commutative algebra in $\mathcal{C}^\otimes$.

Definition 1.3. Let $\mathcal{C}^\otimes$ is a symmetric monoidal ∞-category and $A \in \mathrm{CAlg}(\mathcal{C}^\otimes)$ is an commutative algebra, in this situation we say that A is *idempotent* if the multiplication

$$A \otimes A \xrightarrow{m} A$$

is an equivalence.

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Remark 1.4. Note that any commutative algebra $A \in \mathrm{CAlg}(\mathcal{C}^{\otimes})$ the unit map $e\colon 1 \to A$ is a section of the multiplication map, it follows from two-out-of-three that if A is idempotent, the underlying $\mathbb{E}_0$-algebra is idempotent.

Theorem 1.5. *If $A \in \mathrm{CAlg}(\mathcal{C})$ is idempotent, then the forgetful functor*

$$\mathrm{Mod}_A(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$$

is fully faithful, with essential image $L\mathcal{C}^{\otimes}$.

Proof. If $M \in \mathrm{Mod}_A(\mathcal{C})$, then the composition of maps

$$M \otimes 1 \xrightarrow{M \otimes \eta} M \otimes A \xrightarrow{m} M$$

is homotopic to the identity, hence $M \in L\mathcal{C}$, as $L\mathcal{C}$ is closed under retracts. Now as the map $A \otimes A \xrightarrow{m} A$ is an equivalence, then we see that for all $M \in \mathrm{Mod}_A(\mathcal{C})$ we see that the counit $M \otimes A \to M$ of the free-forgetful adjunction. This follows as the forgetful functor is conservative and we have a commutative square in $\mathcal{C}$

$$\begin{array}{ccc} M \otimes A \otimes A & \xrightarrow{m \otimes A} & M \otimes A \\ \big\downarrow{\scriptstyle M \otimes m} & & \big\downarrow \\ M \otimes A & \xrightarrow{\quad m \quad} & M \end{array}$$

where the left vertical map is an equivalence because A is idempotent, m is an equivalence as $M \in L\mathcal{C}$ and $m \otimes A$ is an equivalence as as m is an equivalence. It thus follows from two-out-of-three that $M \otimes A \to M$ is an equivalence. Hence the forgetful functor $\mathrm{Mod}_A(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ is fully faithful, with essential image $L\mathcal{C}^{\otimes}$. ☐

Example 1.6. The following are examples of idempotents.

(1) If $\mathcal{C}^{\otimes} = \mathrm{Ab}^{\otimes_{\mathbb{Z}}}$, then $\mathbb{Q}$ is an idempotent so the forgetful functor

$$\mathrm{Vect}_{\mathbb{Q}} \to \mathrm{Ab}$$

is fully faithful. In fact the essential image is the uniquely divisible abelian groups.

(2) If $\mathcal{C}^{\otimes} = \mathrm{Pr}^{L,\otimes}$, then $(\mathrm{Sp}, \Sigma^{\infty}_{+})$ is an idempotent. In particular Sp can be promoted to a symmetric monoidal ∞-category, $\mathrm{Sp}^{\otimes}$, where $- \otimes -\colon \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$, preserves colimits in each variable and tensor unit is $\mathbb{S}$, as Ani is the free cocompletion of a point, and Σ^{∞}_{+} corresponds to the sphere spectrum under the equivalence

$$\mathrm{Fun}^{L}(\mathrm{Ani}, \mathrm{Sp}) \xrightarrow{\simeq} \mathrm{Fun}(pt, \mathrm{Sp}) \simeq \mathrm{Sp}.$$

This symmetric monoidal structure, is called the smash product of spectra.

2. Stable Recollement

We will now construct the stable recollement of a reflexive/coreflexive subcategory of a stable ∞-category. For this section we will fix a stable ∞-category $\mathcal{C}$ and a full subcategory $\mathcal{D}$ with inclusion $i_*\colon \mathcal{D} \to \mathcal{C}$. Furthermore, we suppose i_* admits both adjoints, such that we are in the situation

$$\mathcal{C} \; \underset{i^!}{\overset{i^*}{\rightrightarrows}} \; \mathcal{D}, \qquad \mathcal{C} \xleftarrow{\;i_*\;} \mathcal{D}$$

Note that i_* is exact so $\mathcal{D}$ is also stable.

Theorem 2.1. *In this situation, if we let*

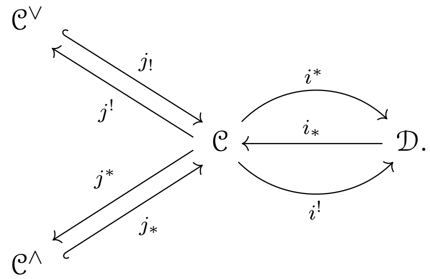
$$j_!\colon \mathcal{C}^{\vee} \hookrightarrow \mathcal{C}$$

denote the inclusion of the full subcategory of $\mathcal{C}$ spanned by M such that $\mathrm{map}(M, i_*(N)) \simeq *$ for all $N \in \mathcal{D}$.

Likewise, let

$$j_* \colon \mathcal{C}^\wedge \hookrightarrow \mathcal{C}$$

be the inclusion of the full subcategory spanned by $M \in \mathcal{C}$ *such that* $\operatorname{map}(i_*(N), M) \simeq *$ *for all* $N \in \mathcal{D}$*. Then there are adjunctions*



Moreover, the composite adjunction

$$\mathcal{C}^\vee \underset{j^! j_*}{\overset{j^* j_!}{\rightleftarrows}} \mathcal{C}^\wedge$$

is an adjoint equivalence.

Proof. We first show that $j_* \colon \mathcal{C}^\wedge \to \mathcal{C}$ admits a left adjoint. The case for $j_!$ is dual. We consider the cofiber sequence

$$i_* i^! \xrightarrow{\varepsilon} \mathrm{id}_{\mathcal{C}} \to \operatorname{cof} \varepsilon$$

in $\operatorname{Fun}(\mathcal{C}, \mathcal{C})$, where ε is the counit of the adjunction $i_* \dashv i^!$. We claim that $\operatorname{cof}\varepsilon$ takes values in $\mathcal{C}^\wedge$. Note that for $Y \in \mathcal{D}$ and $X \in \mathcal{C}$ we have a fiber sequence

$$\operatorname{map}(i_*(Y), i_* i^!(X)) \to \operatorname{map}(i_*(Y), X) \to \operatorname{map}(i_*(Y), \operatorname{cof}\varepsilon(X))$$

of spectra. Thus it suffices to show that $\operatorname{map}(i_*(Y), i_* i^!(X)) \to \operatorname{map}(i_*(Y), X)$ is an equivalence. Now, using the adjunctions

$$\begin{aligned} \operatorname{map}(i_*(Y), i_* i^!(X)) &\simeq \operatorname{map}(Y, i^!(X)) \\ &\simeq \operatorname{map}(i_*(Y), X). \end{aligned}$$

So $\operatorname{map}(i_*(Y), \operatorname{cof}\varepsilon(X)) \simeq *$. Hence, it follows that $\operatorname{cof}\varepsilon \simeq j_* \circ j^*$ for some functor

$$j^* \colon \mathcal{C} \to \mathcal{C}^\wedge.$$

We claim that j^* is left adjoint to j_*. It follows from [**landhighercats**], that it suffices to show that the composite map

$$\operatorname{map}(j^*(X), Z) \to \operatorname{map}(j_* j^*(X), j_*(Z)) \to \operatorname{map}(X, j_*(Z))$$

is an equivalence. Now j_* is fully faithful, so the map

$$\operatorname{map}(j_*(X), Z) \to \operatorname{map}(j_* j^*(X), j_*(Z))$$

is an equivalence. Furthermore, we have a fiber sequence of spectra

$$\operatorname{map}(j_* j^*(X), j_*(Z)) \to \operatorname{map}(X, j_*(Z)) \to \operatorname{map}(i_* i^!(X), j_*(Z))$$

Now $\operatorname{map}(i_* i^!(X), j_*(Z)) \simeq *$ as $i^!(X) \in \mathcal{D}$ and $Z \in \mathcal{C}^\wedge$. Which proves the claim. Note that this implies that j_* is exact, so $\mathcal{C}^\wedge$ is stable.

For the claim that $j^* j_! \dashv j^! j_*$ is an adjoint equivalence, note that the counit $j^* j_! j^! j_* \to \mathrm{id}_{\mathcal{C}^\wedge}$ factors as

$$j^* j_! j^! j_* \xrightarrow{j^* \varepsilon j_*} j^* j_* \xrightarrow{\varepsilon'} \mathrm{id}_{\mathcal{C}^\wedge},$$

with ε the counit of the $j_! \dashv j^!$ adjunction and ε' the counit of the $j^* \dashv j_*$ adjunction. Now ε is an equivalence, as j_* is fully faithful, so it suffices to show that $j^* \varepsilon j_*$ is an equivalence. This follows as we have fiber sequence

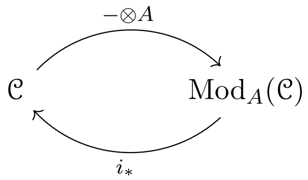
$$j^* j_! j^! j_* \xrightarrow{j^* \varepsilon j_*} j^* j_* \longrightarrow j^* i_* i^* j_*$$

so it suffices to show that $j^*i_*i^*j_* \simeq 0$. We will prove that $j^*i_* \simeq 0$. For this we note that there is a fiber sequence

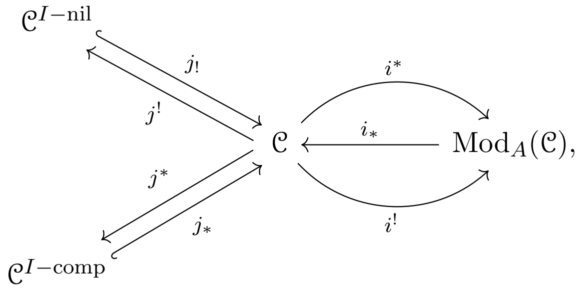
$$i_*i^*i_* \to i_* \to j_*j^*i_*$$

where the first map is an equivalence as i_* is fully faithful. Now j_* is also fully faithful, and thus conservative, so $j_*j^*i_* \simeq 0$ implies that $j^*i_* \simeq 0$. The other composite can be show to be naturally equivalent to the identity in a dual fashion. □

Let us now return to the case presented in section 1. If $\mathcal{C}^{\otimes}$ is a presentably symmetric monoidal stable ∞-category and $\eta\colon 1 \to A$ is an idempotent $\mathbb{E}_0$-algebra, then we showed above that we get an adjunction



Now as the forgetful functor (which is fully faithful as A is idempotent) $i_*\colon \mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ preserves with colimits by [**lurie2017higher**], it follows from the adjoint functor theorem, that i_* admits a right adjoint $i^!$, it follows that we get a stable recollement



where $I \coloneqq \mathrm{fib}(1 \to A)$.

One example of this is the following. Let R be a ring and $x \in R$ a non-zero divisor, in this situation $R \to R[x^{-1}]$ is an idempotent $\mathbb{E}_0$-algebra in $\mathrm{QCoh}(\mathrm{Spec}(R))^{\otimes_R} \simeq \mathcal{D}(R)^{\otimes_R}$, here $j_*\colon \mathrm{QCoh}(\mathrm{Spec}(R))^{I-\mathrm{comp}} \to \mathcal{D}(R)$ identifies with the inclusion of I-complete R-modules and $j_!$ identifies with the inclusion of I-nilpotent modules in $\mathrm{QCoh}(\mathrm{Spec}(R))$.

Studying Ore-localization of $\mathbb{E}_1$-algebras this example holds in greater generality which is proven in [**lurie2017higher**].