THE Q-SHAPED DERIVED CATEGORY

MARIUS VERNER BACH NIELSEN

Abstract. In this thesis we will for any locally Gorenstein category $\mathcal{A}$ with enough projectives and a sufficiently nice ${}_k\text{Mod}$-enriched category Q study the category of Q-shaped modules in $\mathcal{A}$. To this category we associate two model structures, one *projective* and one *injective* named after their trivially cofibrant and trivially fibrant objects, respectively. These model structures have the same weak equivalence and the *Q-shaped derived category* of $\mathcal{A}$ is the Kan localization at this class of maps. We show that under appropriate assumptions on Q and $\mathcal{A}$ there exists cohomology functors, $H^i_{[q]}(-)\colon {}_{Q,\mathcal{A}}\text{Mod} \to \mathcal{A}$, such that a map is a weak equivalence if and only if it induces an isomorphism on cohomology for every $i > 0$ and $q \in Q$. All of the above is heavily based on ideas from [HJ21], and in the final two chapters we apply our results to recover most of the results of Holm and Jørgensens paper.

Contents

Supervised by Henrik Holm
Date: *June 8, 2022.*

1. Introduction

This is my masters thesis at University of Copenhagen in 2022. The subject of my thesis is '*The Q-shaped derived category*' and is based on a paper by H. Holm and P. Jørgensen [HJ21] under the name '*The Q-shaped derived category of a ring*' and the purpose of the thesis is read and report on the results of their paper. With this said we also set out to improve on their results and we have succeded to some degree. Throughout this thesis we will try to make it explicit where we have had a substantial different approach than Holm and Jørgensen. We will also make it explicit in this introduction what our results are and how they compare to the results of Holm and Jørgensen.

If $\mathcal{A}$ is an abelian category, then we may consider its category of chain complexes $\mathrm{Ch}(\mathcal{A})$ in $\mathcal{A}$. If $\mathcal{A}$ is bicomplete and has enough projectives respectively injectives, then $\mathrm{Ch}(\mathcal{A})$ admits *projective* respectively *injective* model structure, in which the weak equivalences are the quasi-isomorphisms.

The category of chain complexes is equivalent to the category of additive functors from the Ab-enriched category Q with objects $\mathrm{Ob}(Q) = \mathbb{N}$ and maps given by

$$Q(p, q) = \begin{cases} \mathbb{Z} & \text{if } q = p \text{ or } q = p - 1 \\ 0 & \text{else} \end{cases}$$

where we denote the generator of $Q(p, p-1)$ by ∂_p and the generator of $Q(p, p)$ by id_p.

The main results of [HJ21] states that for any ring R and small Ab-enriched category Q, which is similar to the indexing category in the case of chain complexes[1], then on the category of Q-shaped modules in ${}_R\mathrm{Mod}$, that is ${}_{Q,R}\mathrm{Mod} := \mathrm{Fun}^{\mathrm{add}}(Q, {}_R\mathrm{Mod})$, there are *projective* and *injective* model structures and cohomology functors

$$H^i_{[q]}(-) \colon {}_{Q,R}\mathrm{Mod} \to {}_R\mathrm{Mod}$$

for all $i > 0$ and $q \in Q$. Such that the weak equivalences in these two model structures coincide and a map $f \colon X \to Y$ is a weak equivalence if and only if f is a "quasi-isomorphism" with respect to these cohomology functors. That is

$$H^i_{[q]}(f) \colon H^i_{[q]}(X) \to H^i_{[q]}(Y)$$

is an isomorphism for all $i > 0$ and $q \in Q$.

A very rough sketch of their argument is as follows. First, prove the theorem for the case of k a commutative Gorenstein ring of homological dimension 1, and then lift the result to all k-algebras, A.

In this thesis we will enlarge the class of categories for which first step of the proof in [HJ21] holds. That is, in section 4, 5 and 6, we prove the following statements.

Theorem (Theorem 5.1 and Theorem 5.7)**.** *If Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3 and $\mathcal{A}$ is locally Gorenstein category with enough projectives, then the following hold.*

(1) There exists an abelian model structure on ${}_{Q,\mathcal{A}}\mathrm{Mod}$, with cofibrant objects given by the Gorenstein projective objects, trivial objects given by the objects with finite injective dimension and every object fibrant.

(2) There exists an abelian model structure on ${}_{Q,\mathcal{A}}\mathrm{Mod}$ with every object cofibrant, the trivial objects given by the objects with finite projective dimension and fibrant objects given by the Gorenstein injective objects.

Furthermore, the homotopy categories coincide and there is a canonical choice of triangulated structure on $\mathcal{D}_Q(\mathcal{A})$.

In the case where Q satisfies Setup 4.3* there is an ideal in Q called the *pseudo-radical* which we denote by τ. Similarly to the authors of [HJ21] we leverage this ideal to construct cohomology functors

$$H^i_{[q]}(-) \colon {}_{Q,\mathcal{A}}\mathrm{Mod} \to \mathcal{A}$$

for all $i > 0$ and $q \in Q$, such that the following theorems hold.

[1]The technical condition is that Q satisfies Setup 4.3*.

Theorem (Theorem 6.20)**.** *If Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3* such that the pseudo-radical of Q is nilpotent and $\mathcal{A}$ is a locally 1-Gorenstein category with enough projectives, then for any Q-shaped module $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$ in $\mathcal{A}$ the following are equivalent*

(1) The object X is trivial, that is $X \to 0$ is a weak equivalence.
(2) For all $i > 0$ and $q \in Q$ we have that $H^i_{[q]}(X) \cong 0$.
(3) For all $q \in Q$ we have that $H^1_{[q]}(X) \cong 0$

Theorem (Theorem 6.21)**.** *If Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3* such that the pseudo-radical of Q is nilpotent and $\mathcal{A}$ is a locally Gorenstein category with enough projectives and global dimension 1, then for any map $f\colon X \to Y$ of Q-shaped modules in $\mathcal{A}$ the following are equivalent*

(1) the map $f\colon X \to Y$ is a weak equivalence in the either the projective or injective model structure.
(2) The map $H^i_{[q]}(f)$ is an isomorphism for all $i > 0$ and $q \in Q$.
(3) The map $H^i_{[q]}(f)$ is an isomorphism for all $i \in \{1, 2\}$ and $q \in Q$.

Comparing to the case of chain complexes the first theorem states that a Q-shaped module is "exact" if and only if its cohomology vanishes, and the second states that a map is a weak equivalence if and only if it is a "quasi-isomorphism" on cohomology.

Afterwards, we will discuss the second step the in proof. This will be done in the setting used in [HJ21]. That is, we set $\mathcal{A} = {}_k\mathrm{Mod}$, for k a Gorenstein ring, and show that in this setting we can lift our results to the category of Q-shaped A-modules for all k-algebras, A, where A has finite injective dimension as a k-module. This part will be covered in section 7 of the thesis.

The final part of this thesis will be a quick exposition of section 8 in [HJ21], in which the authors apply the theory developed to examples coming from the representation theory of quivers. This will be covered in section 8 of the thesis.

Prerequisites. This thesis is written with a strong background of abstract homotopy in mind. In particular, we will assume the reader is familiar with Quillen's notion of model categories. The books [Hir09], [Hov07] and [Lur09] are good references. We will also assume that the reader has extensive knowledge of category theory, as found by reading [Mac13], [Lor15] and [Kel82]. We will not use the full strength of enriched-category theory, as we will only consider categories enriched in ${}_R\mathrm{Mod}$, where R is a commutative ring. In this case the theory simplifies quite a lot, and we will only consider conical (co)limits and simple (co)end constructions. Therefore, we feel confident that even readers with little to no knowledge of enriched categories can read this thesis without problems. Finally, we assume the reader is familiar with homological algebra as covered in [Wei95].

Acknowledgements. First and foremost I would like to thank my supervisor Henrik Holm, for introducing me to this interesting subject, our very enjoyable meetings and his support throughout the writing of this thesis.

Secondly, I would like to thank Marius and Oscar, my fellow students from the math department, for there thorough questioning of my results and our discussion, when I presented to them on the subject of my thesis.

Thirdly, I would like to thank my girlfriend Mai for her support throughout these last few months. I feel so lucky to have you in my life.

Lastly, I would like to thank my family for their support and in particular my grandfather Poul, who we unfortunately lost while I was writing this thesis. May you rest in peace.

Conventions and notation. Throughout this thesis k will always denote a commutative ring and Q will always be a small category enriched in $({}_k\mathrm{Mod}, - \otimes_k -, k)$. In order to make it easier to differentiate between maps in the domain and maps in the target of our functor categories we will denote the k-module of maps from p to q by $Q(p, q)$ and use the $\mathrm{Hom}(-, -)$ notation otherwise. If $\mathcal{A}$ is a ${}_k\mathrm{Mod}$-enriched category we will denote the category of k-linear functors from

Q to $\mathcal{A}$ by $\mathrm{Fun}^k(Q,\mathcal{A})$, this category is naturally the underlying category of a ${}_k\mathrm{Mod}$-enriched category with same objects, and mapping modules given by the end

$$\mathrm{Hom}_{Q,\mathcal{A}}(X,Y) := \int_{q\in Q} \mathrm{Hom}_{\mathcal{A}}(X(q),Y(q)).$$

Note that since k is free on one variable as a k-module, the underlying set will be the same as the set of natural transformations from X to Y, and as so we will not distinguish between the two.

Whenever we write an adjunction, we will always put the left adjoint on top, if we write a triple adjunction we stick to this convention. In particular, by this we indicate that the middle functor both a left adjoint and a right adjoint functor.

We will often consider the category $\mathrm{Fun}^k(Q,\mathcal{A})$ as Q-shaped modules in $\mathcal{A}$, the reasoning behind this is given in the beginning of section 3. In accordance to this we will write

$${}_{Q,\mathcal{A}}\mathrm{Mod} := \mathrm{Fun}^k(Q,\mathcal{A})$$

for the category of (left) Q-shaped modules in $\mathcal{A}$. When R is a k-algebra we will abuse notation further and write ${}_{Q,R}\mathrm{Mod}$ for the category of Q-shaped modules in ${}_R\mathrm{Mod}$ and if $R = k$, we will write ${}_Q\mathrm{Mod}$. We will adopt similar naming schemes for right Q-shaped modules in $\mathcal{A}$, which we denote by $\mathrm{Mod}_{Q,\mathcal{A}} := \mathrm{Fun}^k(Q^{\mathrm{op},\mathcal{A}})$. Finally, we will denote the full subcategory of ${}_{Q,\mathcal{A}}\mathrm{Mod}$ spanned by (Gorenstein) projective, respectively (Gorenstein) injective, respectively modules with finite projective dimension by $(\mathrm{G})\mathrm{Prj}_{Q,\mathcal{A}}$, $(\mathrm{G})\mathrm{Inj}_{Q,\mathcal{A}}$ and $\mathcal{L}_{Q,\mathcal{A}}$ respectively. W

2. Preliminaries

In this section we will recall the preliminary notions which will be used for the entirety of this thesis. We will start with by recalling the basic notions in the theory of $({}_k\mathrm{Mod}, \otimes_k, k)$-enriched categories. We will abuse notation and just refer to these as ${}_k\mathrm{Mod}$-enriched categories.

Definition 2.1. A ${}_k\mathrm{Mod}$-enriched category $\mathcal{A}$ is *k-linear* if its underlying category $\mathcal{A}_0$ is abelian. That is

(1) $\mathcal{A}$ admits finite products and coproducts and the canonical comparison map is an isomorphism in $\mathcal{A}$.
(2) $\mathcal{A}$ admits all kernels and cokernels
(3) Every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

Remark 2.2. Note that normally one requires $\mathcal{A}$ to be pointed. This however is redundant, given the first axiom, as an initial object is simply a nullary coproduct. Likewise, a terminal object is simply a nullary product and the canonical map is an isomorphism by the additivity assumption.

Remark 2.3. Note that for any commutative ring k the forgetful functor

$${}_k\mathrm{Mod} \xrightarrow{i^*} \mathrm{Ab}$$

is conservative and admits both adjoints. In particular, it creates and preserves both limits and colimits. It follows, that it makes no difference to ask the limits, respectively colimits, mentioned in Definition 2.1 to be limits, respectively colimits in $\mathcal{A}$ rather than $\mathcal{A}_0$.

Definition 2.4. A k-linear category $\mathcal{A}$ is *Grothendieck* if:

(1) it admits all coproducts.
(2) Filtered colimits of exact sequences are exact.
(3) There exists a generator.

We will now recall a few basic results on Grothendieck categories.

Theorem 2.5. *Let $\mathcal{A}$ be a Grothendieck k-linear category. In this situation the following hold:*

(1) The category $\mathcal{A}$ is complete.
(2) The category $\mathcal{A}$ admits a injective cogenerator.

(3) The category $\mathcal{A}$ has enough injectives.

Proof. Statement (1) follows from Gabriel-Popescu's theorem [KS05, Thm. 8.5.8], (2) follows from (3) and (3) is proven in [KS05, Thm. 9.6.3]. □

Remark 2.6. Note that if $\mathcal{A}$ is any k-linear category with all coproducts, then $\mathcal{A}$ is tensored over ${}_k\text{Mod}$ in the following way. For $k^{\oplus S} \in {}_k\text{Mod}$ and $X \in \mathcal{A}$ we define

$$k^{\oplus S} \otimes X := \bigoplus_S X.$$

Now if $M \in {}_k\text{Mod}$ is any k-module, with presentation

$$k^{\oplus R} \to k^{\oplus G} \to M \to 0$$

then we set

$$M \otimes X := \operatorname{cok}\left(k^{\oplus R} \otimes X \to k^{\oplus G} \otimes X\right).$$

Likewise, one can see that $\mathcal{A}$ is cotensored over ${}_k\text{Mod}$ if $\mathcal{A}$ admits all products.

Notation. If $\mathcal{A}$ is any (co)complete k-linear category, we will let $M \otimes X$ denote the tensor of $X \in \mathcal{A}$ with $M \in {}_k\text{Mod}$ and let X^M denote the cotensor.

Lemma 2.7. *For $- \otimes -\colon {}_k\text{Mod} \times \mathcal{A} \to \mathcal{A}$ as above, and $\mathcal{A}$ a Grothendieck k-linear category, it holds that*

(1) if $P \in {}_k\text{Mod}$ is projective, then the functor $P \otimes -\colon \mathcal{A} \to \mathcal{A}$ preserves projectives.
(2) If $P \in {}_k\text{Mod}$ is projective, then the functor $P \otimes -\colon \mathcal{A} \to \mathcal{A}$ is exact.
(3) If $P \in {}_k\text{Mod}$ is finitely generated and projective, then $\operatorname{Hom}_k(P, k) \otimes - \cong (-)^P$.

Proof. Ad (1) for $P \in {}_k\text{Mod}$ and $X \in \mathcal{A}$ projective, we want to see $\operatorname{Hom}_{\mathcal{A}}(P \otimes X, -)$ is exact. This follows as

$$\operatorname{Hom}_{\mathcal{A}}(P \otimes X, -) \cong \operatorname{Hom}_k(P, \operatorname{Hom}(X, -)).$$

Which is the composite of exact functors and thus exact.

Ad (2) suppose $F \in {}_k\text{Mod}$ is free on a set S, then we have $F \otimes X \cong \oplus_S X$ for all $X \in \mathcal{A}$. Which is exact, as $\mathcal{A}$ is Grothendieck. It follows that the claim holds for projective k-modules, as any projective module is a direct summand of a free module.

Ad (3) if $k^n \in {}_k\text{Mod}$ is free of finite rank, then

$$\operatorname{Hom}_k(k^n, k) \otimes X \cong k^n \otimes X \cong \oplus_n X \cong X^n \cong X^{k^n}.$$

The result now follows for finitely generated projective modules, as they all are summands of free modules of finite rank. □

Definition 2.8. Let $\mathcal{A}$ be a k-linear category, an object $X \in \mathcal{A}$ is *compact* if the functor

$$\operatorname{Hom}_{\mathcal{A}}(X, -)\colon \mathcal{A} \to {}_k\text{Mod}$$

preserves filtered colimits.

Definition 2.9. Let $\mathcal{A}$ be a k-linear category, a short exact sequence

$$0 \to X' \to X \to X'' \to 0$$

is said to be *pure* if

$$0 \to \operatorname{Hom}_{\mathcal{A}}(C, X') \to \operatorname{Hom}_{\mathcal{A}}(C, X) \to \operatorname{Hom}_{\mathcal{A}}(C, X'') \to 0$$

is exact for all compact $C \in \mathcal{A}$.

In this thesis we will be particularly interested in a full subcategory of $\mathcal{A}$ containing the projective objects, namely the Gorenstein projective object. We will now briefly recall the definition of Gorenstein projectives and a characterization in the category ${}_R\text{Mod}$ for a ring R.

Definition 2.10. Let $\mathcal{A}$ be a k-linear category. An object $X \in \mathcal{A}$ is *Gorenstein projective* if there exists an exact chain complex $P_\bullet$ with $X \cong \ker(d_0)$ and $P_i \in \mathcal{A}$ projective for all $i \in \mathbb{Z}$. Such that for any projective $Q \in \mathcal{A}$ the complex $\operatorname{Hom}_{\mathcal{A}}(P_\bullet, Q)$ is exact. We denote the full subcategory spanned by Gorenstein projectives by $\text{GPrj}_{\mathcal{A}}$.

It is clear that any projective object is Gorenstein projective. However, it is not immediately clear that these two classes might differ.

Example 2.11. Let K be a field and consider the dual numbers of K

$$R := K[x]/(x^2)$$

and the complex

$$P_\bullet := \ldots \to R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \to \ldots$$

this is clearly exact in ${}_R\mathrm{Mod}$ and for any R-module M we have that

$$\mathrm{Hom}_R(P_\bullet, M) \cong \ldots \to M \xrightarrow{x\cdot} M \xrightarrow{x\cdot} M \to \ldots$$

which is exact for same reason as above. Now the kernel $\ker(d_0) \cong \mathrm{im}(d_{-1}) \cong R/(x) \cong K$ is thus a Gorenstein projective R-module. However K has non-trivial x-torsion, and therefore is not flat, and hence not projective.

Definition 2.12. Let $\mathcal{A}$ be a k-linear category. An object $X \in \mathcal{A}$ is *Gorenstein injective* if X is Gorenstein projective in $\mathcal{A}^{\mathrm{op}}$.

Definition 2.13. Let $\mathcal{A}$ be a k-linear category. For an object $X \in \mathcal{A}$ we define the *Gorestein projective dimension* of X to be

$$\mathrm{Gpd}_{\mathcal{A}} X := \inf\{n \in \mathbb{N} \mid \text{exists a resolution } G_\bullet \text{ of } X \text{ such that } G_i \in \mathrm{GPrj}_{\mathcal{A}} \text{ and } G_i = 0 \text{ for all } i \geqslant n\}.$$

The *Gorenstein injective dimension* of X is the Gorenstein projective dimension of X as an object of $\mathcal{A}^{\mathrm{op}}$. Moreover, the *global Gorenstein projective dimension* of $\mathcal{A}$ is defined as

$$\mathrm{glGpd}\,\mathcal{A} = \sup\{n \in \mathbb{N} \mid \exists X \in \mathcal{A} : \mathrm{Gpd}_{\mathcal{A}} X = n\}.$$

The *global Gorenstein injective dimension* is defined analogously.

Lemma 2.14. *If $\mathcal{A}$ is a k-linear category with enough projectives, then for all $X \in \mathcal{A}$ we have that*

$$\mathrm{Gpd}_{\mathcal{A}} X \leqslant \mathrm{pd}_{\mathcal{A}} X.$$

Furthermore, it follows that $\mathrm{glGpd}\,\mathcal{A} \leqslant \mathrm{gldim}\,\mathcal{A}$.

Proof. Suppose, $P_\bullet \to X$ is a projective resolution of X, then since P_i is projective for all $i \geqslant 0$ we know that P_i is Gorenstein projective for all $i \geqslant 0$. Therefore follows that $P_\bullet \to X$ is a Gorenstein projective resolution of X. It follows that

$$\mathrm{Gpd}_{\mathcal{A}} X \leqslant \mathrm{pd}_{\mathcal{A}} X.$$

The furthermore part, follows directly from the definitions from the above. □

We will now move on to recall the basic notions of cotorsion pairs, and Hovey's theorem on abelian model categories.

Definition 2.15. Let $\mathcal{A}$ be an k-linear category. If $\mathcal{C}$ is a full subcategory of $\mathcal{A}$, then we define the *right orthogonal complement* of $\mathcal{C}$ to be the full subcategory of $\mathcal{A}$, denoted $\mathcal{C}^\perp$, spanned by objects Y such that for all $X \in \mathcal{C}$ we have that

$$\mathrm{Ext}^1_{\mathcal{A}}(X, Y) \cong 0.$$

Similarly, we define the *left orthogonal complement* of $\mathcal{C}$, denoted ${}^\perp\mathcal{C}$, to be the full subcategory of $\mathcal{A}$ spanned by objects $Y \in \mathcal{A}$ such that for all $X \in \mathcal{C}$ we have that

$$\mathrm{Ext}^1_{\mathcal{A}}(Y, X) \cong 0.$$

A pair of full subcategories $(\mathcal{C}, \mathcal{D})$ is called a *cotorsion pair* if $\mathcal{C}^\perp = \mathcal{D}$ and ${}^\perp\mathcal{D} = \mathcal{C}$.

Definition 2.16. If $\mathcal{A}$ is a k-linear category and $(\mathcal{C}, \mathcal{D})$ is a cotorsion pair. Then we say that $(\mathcal{C}, \mathcal{D})$ *has enough projectives* if for all $X \in \mathcal{A}$ there is a short exact sequence

$$0 \to Z \to Y \to X \to 0$$

with $Y \in \mathcal{C}$ and $Z \in \mathcal{D}$.

Dually we say that $(\mathcal{C}, \mathcal{D})$ *has enough injectives* if for all $X \in \mathcal{A}$ there is a short exact sequence

$$0 \to X \to Y \to Z \to 0$$

with $Y \in \mathcal{D}$ and $Z \in \mathcal{C}$.

We say that a cotorsion pair $(\mathcal{C}, \mathcal{D})$ is *complete* if it has enough projectives and has enough injectives.

Example 2.17. Here are two examples of cotorsion pairs

(1) If $\mathcal{A}$ is a k-linear category, then $(\mathrm{Prj}_{\mathcal{A}}, \mathcal{A})$ is a cotorsion pair. In this case $(\mathrm{Prj}_{\mathcal{A}}, \mathcal{A})$ is complete if and only if $\mathcal{A}$ has enough projectives.
(2) If R is a ring and $\mathcal{F}$ is the class of flat left R-modules, then $(\mathcal{F}, \mathcal{F}^{\perp})$ is a cotorsion pair. The class $\mathcal{F}^{\perp}$ is called the cotorsion modules.

Definition 2.18. Let $\mathcal{A}$ be a k-linear category and $(\mathcal{C}, \mathcal{D})$ a cotorsion pair in $\mathcal{A}$. In this situation we say that $(\mathcal{C}, \mathcal{D})$ is *hereditary* if

$$\mathrm{Ext}^i_{\mathcal{A}}(X, Y) \cong 0$$

for all $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and $i > 0$.

Additionally, we say that $(\mathcal{C}, \mathcal{D})$ is *resolving* if $\mathcal{C}$ is closed under kernels of epimorphisms and *coresolving* if $\mathcal{D}$ is closed under cokernels of monomorphisms.

Remark 2.19. It is clear that if a cotorsion pair is hereditary, then it is also resolving and coresolving. The converse that a cotorsion pair which is resolving, respectively coresolving is hereditary holds if the category has enough projectives and injectives, respectively.

The reason we are interested in cotorsion pairs in this thesis is because of their connection to model structures on abelian categories.

Definition 2.20. Let $\mathcal{A}$ be a bicomplete k-linear with a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. We say that $\mathcal{A}$ is an *abelian* model category if the following hold:

(1) A map $f : X \to Y$ in $\mathcal{A}$ is a (trivial) cofibration if and only if it is a monomorphism with (trivially) cofibrant cokernel.
(2) A map $f : X \to Y$ in $\mathcal{A}$ is a (trivial) fibration if and only if it is is a epimorphism with (trivially) fibrant kernel.

Hovey proves in [Hov02] that for any abelian model category, the model structure is intrinsically determined by the full subcategories spanned by of cofibrant, fibrant and trivial objects respectively. These three subcategories then determine two cotorsion pairs. It turns out that such cotorsion pairs, determine abelian model structures and this corresponds is bijective. The correspondence is explained by the following two theorems.

Theorem 2.21 (Prop. 2.2 & Lemma 2.4 [Hov02])**.** *Suppose $\mathcal{A}$ is an abelian model category. Let $\mathcal{C}$ denote the full subcategory of cofibrant objects, $\mathcal{F}$ the full subcategory of fibrant objects and $\mathcal{W}$ the full subcategory of trivial objects, then*

(1) The class $\mathcal{W}$ is thick. That is, it is closed under retracts, and if of two of the three entries in a short exact sequence are in $\mathcal{W}$, then so is the third.
(2) The pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are complete cotorsion pairs.

Theorem 2.22 (Thm. 2.5 [Hov02])**.** *If $\mathcal{A}$ is a bicomplete k-linear category and $\mathcal{C}$, $\mathcal{W}$ and $\mathcal{F}$ are full subcategories of $\mathcal{C}$ such that*

(1) $\mathcal{W}$ is thick.
(2) The pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are complete cotorsion pairs.

Then there exists a unique abelian model structure on $\mathcal{A}$ such that $\mathcal{C}$ is the full subcategory spanned by cofibrant objects, $\mathcal{W}$ the full subcategory of trivial objects and $\mathcal{F}$ is the full subcategory of fibrant objects.

Definition 2.23. Let $\mathcal{A}$ be a bicomplete k-linear category, a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ satisfying the conditions of Theorem 2.22 is called a *Hovey triple.*

Example 2.24. Here are two examples of abelian model categories.

(1) If $\mathcal{A}$ is a Grothendieck k-linear category, then the the injective model structure on $\mathrm{Ch}(\mathcal{A})$ is determined by the Hovey triple $(\mathrm{Ch}(\mathcal{A}), \mathrm{Ex}, \mathrm{DG}-\mathrm{Inj})$, where Ex is the full subcategory spanned by exact chain complexes and $\mathrm{DG}-\mathrm{Inj}$ is the full subcategory spanned by DG-injective complexes. That is the complexes with injective entries, such that any map from an exact complex is null homotopic.
(2) If R is a Gorenstein ring, then the stable module model structure on ${}_R\mathrm{Mod}$ is determined by the Hovey triple $(\mathrm{GPrj}_R, \mathcal{L}_R, {}_R\mathrm{Mod})$.

In both of these cases the resulting homotopy categories are triangulated. This turns out not to be an accident.

Theorem 2.25 ([Gil16, Thm.4.3])**.** *If $\mathcal{A}$ is an abelian model category, with Hovey triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ such that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are complete hereditary cotorsion pairs. Then there exists a triangulated structure on* $\mathrm{Ho}(\mathcal{A})$*, where the suspension functor is determined on X by taking a short exact sequence*

$$0 \to X \to W \to \Sigma X \to 0$$

where $W \in \mathcal{W}$.

Remark 2.26. In fact if $\mathcal{A}$ is a hereditary abelian model category, then the full subcategory spanned by bifibrant objects is a Frobenius category and the stable category on $\mathcal{C} \cap \mathcal{F}$ is equivalent to $\mathrm{Ho}(\mathcal{C})$ by a fundamental theorem on model categories. The above triangulated structure is constructed exactly such that this equivalence is a triangulated equivalence.

Subobjects, intersections and preimages. We will now recall give a short discussion on subobjects, quotients and intersections in any bicomplete abelian category. This will be relevant in section 6, where some arguments are more easily phrased in terms of these. The main results of this subsection, will show that with only a little extra care, these act exactly as in the case of modules over a ring.

For the rest of this section $\mathcal{A}$ will be a bicomplete k-linear category.

Definition 2.27. For an object $X \in \mathcal{A}$ a *subobject* of X is a monomorphism $\alpha \colon Y \to X$. We say that (Y, α) is a subobject of X. A map of subobjects $(Y, \alpha) \to (Z, \beta)$ is a map $f \colon Y \to Z$ of objects over X. We denote the category of subobjects of X by $\mathrm{Sub}_{\mathcal{A}} X$. Finally, we say that (Y, α) and (Z, β) *represent* the same subobjects of X if they are isomorphic in $\mathrm{Sub}_{\mathcal{A}} X$.

Remark 2.28. Note that any map $f \colon (Y, \alpha) \to (Z, \beta)$ of subobjects the underlying map is a monomorphism in $\mathcal{A}$. This follows as $\alpha = \beta f$ is a monomorphism, so f is a monomorphism. In particular (Y, f) is a subobject of Z.

Note that by definition $\mathrm{Sub}_X \mathcal{A}$ is a full subcategory of $\mathcal{A}/_X$, as so it comes with two functorialities. If $f \colon X \to Y$ is a monomorphism, then there is a functor

$$f_* \colon \mathrm{Sub}_{\mathcal{A}} X \to \mathrm{Sub}_{\mathcal{A}} Y$$

given by restricting $f_* \colon \mathcal{A}/_X \to \mathcal{A}/_Y$ to $\mathrm{Sub}_{\mathcal{A}} X$. There also is a forgetful functor

$$U \colon \mathrm{Sub}_{\mathcal{A}} X \to \mathcal{A}$$

given by sending a subobject $\alpha \colon Y \to X$ to Y and a map $f \colon (Y, \alpha) \to (Z, \beta)$ to $f \colon Y \to Z$.

Definition 2.29. If $\alpha \colon Y \to X$ is a subobject of X, then we define the *quotient* of X by Y to be

$$X/Y := \mathrm{coker}(\alpha \colon Y \to X).$$

Proposition 2.30. *If $X \in \mathcal{A}$ is any object, then the association $(Y, \alpha) \mapsto X/Y$ is functorial for any subobject (Y, α).*

Proof. Let S be span category, $\mathbf{1}$ be the walking arrow category and $i\colon \mathbf{1} \to S$ be the functor inclusion of the walking arrow into the right side of the span. In this situation we may consider the composition of functors

$$\mathrm{Sub}_{\mathcal{A}} X \hookrightarrow \mathcal{A}/_X \to \mathrm{Fun}(\mathbf{1}, \mathcal{A}) \xrightarrow{i_!} \mathrm{Fun}(S, \mathcal{A}) \xrightarrow{\mathrm{colim}} \mathcal{A}$$

and denote it by $X/_{-}$. Then we have

$$X/_Y := \mathrm{colim} \begin{pmatrix} Y & \xrightarrow{\alpha} & X \\ \downarrow & & \\ 0 & & \end{pmatrix} \cong \mathrm{coker}\, \alpha.$$

Which is what we wanted. □

Note that by the above result if (Y, α) and (Z, β) represent the same subobjects of X, then we have $X/Y \cong X/Z$. From this and the 5-lemma it is easy to see that the converse also holds. That is $X/_{-}$ is conservative. Furthermore, we have the following transitive property for quotients.

Proposition 2.31 (3rd isomorphism theorem)**.** *For $X \in \mathcal{A}$, if (Y, α) is a subobject of X and (Z, β) is a subobject of Y, then there is a map $\phi\colon Y/Z \to X/Z$ such that $(Y/Z, \phi)$ is a subobject of X/Z. Furthermore, $(X/Z)/(Y/Z) \cong X/Y$.*

Proof. Note that $Z \xrightarrow{\beta} Y \xrightarrow{\alpha} X$ is a mono so $(Z, \alpha\beta)$ is a subobject of X. Now consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \xrightarrow{\beta} & Y & \longrightarrow & Y/Z & \longrightarrow & 0 \\ & & {\scriptstyle \alpha\beta}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\ 0 & \longrightarrow & X & \xrightarrow[\mathrm{id}_X]{} & X & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

This has exact rows by assumption. Hence, it follows from the snake lemma that we get a short exact sequence

$$0 \to Y/Z \to X/Z \to X/Y \to 0$$

which proves the claim. □

We will now move on to define intersections, before we go to preimages in this setting.

Definition 2.32. For $X \in \mathcal{A}$. Let $(Y_i, \alpha_i)_{i \in I}$ be a collection of subobjects of X, we define the *intersection*, denoted $\bigcap_{i \in I} Y_i$, of $(Y_i, \alpha_i)_{i \in I}$ to be the limit of the diagram $\phi\colon I^{\triangleright} \to \mathcal{A}$ with $\phi(i) = Y_i$ for all $i \in I$ and $\phi(\infty) = X$ and for the unique map $\hat{i}\colon i \to \infty$ we set $\phi(\hat{i}) = \alpha_i$.[2]

Proposition 2.33. *For all sets of subobjects $(Y_i, \alpha_i)_{i \in I}$ the canonical map*

$$\iota\colon \bigcap_{i \in I} Y_i \to X$$

is a monomorphism. We conclude that $(\cap_{i \in I} Y_i, \iota)$ is a subobject of X.

Proof. To see that this map is a monomorphism note that $I^{\triangleright}$ is connected, so if

$$\Delta\colon \mathcal{A} \to \mathrm{Fun}(I^{\triangleright}, \mathcal{A})$$

denotes the constant functor, then we have that

$$\lim_{i \in I^{\triangleright}} \Delta(X) \cong X.$$

[2]Here $I^{\triangleright}$ is the category obtained from I, considered as a discrete groupoid, by adjoining a terminal object. Concretely this can be described as the join $I^{\triangleright} = I \bigstar \Delta^0$, where Δ^0 is the one point category.

The map $\iota\colon \cap_{i\in I} Y_i \to X$ is induced by the natural transformation

$$\eta\colon \phi \to \Delta(X)$$

with components $\eta_i = \alpha_i$ for $i \in I$ and $\eta_\infty = \mathrm{id}_X$, thus it follows from functoriality and the fact that limits commute with limits that

$$\ker \iota \cong \lim_{i\in I^{\triangleright}} (\ker \eta)\,.$$

Now (Y_i, α_i) is a subobject of X for all $i \in I$, so we have that $\ker \eta$ is constant with value 0. It follows, again because $I^{\triangleright}$ is connected, that

$$\ker \iota \cong 0.$$

□

Remark 2.34. Note that the intersection $\cap_{i\in I} Y_i$ is the product of $(Y_i, \alpha_i)_{i\in I}$ considered as objects of $\mathcal{A}/_X$, and therefore in $\mathrm{Sub}_{\mathcal{A}} X$. We can leverage this to recover the following familiar statements

(1) If $(Y_i, \alpha_i)_{i\in I}$ is a collection of subobjects of X, then for all $j \in I$

$$\left(\bigcap_{i\in I\setminus\{j\}} Y_i\right) \cap Y_j \cong \bigcap_{i\in I} Y_i.$$

(2) If $(Y_i, \alpha_i)_{i\in I}$ is a collection of subobjects of X, then

$$\left(\bigcap_{i\in I} Y_i\right) \cap X \cong \bigcap_{i\in I} Y_i$$

(3) If (Y, α) and (Z, β) are subobjects of X, then

$$Y \cap Z \cong Z \cap Y.$$

Definition 2.35. If (Y, α) is a subobject of X and $f\colon Z \to X$ is a map in $\mathcal{A}$, then the *preimage* of Y along f is the pullback

$$\begin{array}{ccc} f^{-1}(Y) & \hookrightarrow & Z \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y & \overset{}{\underset{\alpha}{\hookrightarrow}} & X. \end{array}$$

Proposition 2.36. *Suppose $(Y_i, \alpha_i)_{i\in I}$ is a collection of subobjects of X. Then the canonical map*

$$\bigcap_{i\in I} f^{-1}(Y_i) \to f^{-1}\big(\bigcap_{i\in I} Y_i\big)$$

is an isomorphism of subobjects of Z.

Proof. This is an application of the classical fact that limits preserve limits and the same type of manipulation of constant diagrams which we did in Proposition 2.33. □

Proposition 2.37. *Suppose (Y, α) is a subobject of X, (Y', α') is a subobject of X' and*

$$\bar{f}\colon X/Y \to X'/Y'$$

is a map induced by a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\alpha} & X & \longrightarrow & X/Y \\ \downarrow{\scriptstyle f|_Y} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ Y' & \xrightarrow{\alpha'} & X' & \longrightarrow & X'/Y'. \end{array}$$

In this situation we have that

$$\ker \bar{f} \cong f^{-1}(Y')/Y$$

as subobjects of X/Y.

Proof. By the universal property of pullbacks there is a commutative diagram

$$\begin{array}{ccccc} Y & & \xrightarrow{\quad i \quad} & & \\ & \searrow^{j} & & & \\ & & f^{-1}(Y') & \longrightarrow & X \\ & & \downarrow & & \downarrow f \\ & & Y' & \longrightarrow & X'. \end{array}$$

Where j is the unique map making this diagram commute. Furthermore, since i is a monomorphism it follows that j is a monomorphism. It follows from commutativity of this diagram and functoriality of cokernels that the composite of

$$X/Y \to X/f^{-1}(Y') \xrightarrow{\tilde{f}} X'/Y'$$

is equal to $\bar{f}$. Now consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^{-1}(Y')/Y & \longrightarrow & X/Y & \longrightarrow & X/f^{-1}(Y') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \bar{f} & & \downarrow \tilde{f} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & X'/Y' & \xrightarrow{\text{id}} & X'/Y' & \longrightarrow & 0 \end{array}$$

where the right most square is commutative by the above argument. The top row is exact by the 3rd isomorphism theorem and the bottom row is exact by definition. Finally, the left most square commutes because of the fact that

$$\begin{array}{ccccc} Y & \xrightarrow{\text{id}} & Y & \longrightarrow & Y' \\ \downarrow j & & \downarrow i & & \downarrow \\ f^{-1}(Y') & \longrightarrow & X & \xrightarrow{f} & X' \end{array}$$

commutes and the map $f^{-1}(Y') \to X \xrightarrow{f} X'$ factors through Y'. Hence, the induced map on cokernels is 0. Using the snake lemma we get an exact sequence

$$0 \to f^{-1}(Y')/Y \to \ker \bar{f} \to \ker \tilde{f} \to 0 \to \operatorname{coker} \bar{f} \to \operatorname{coker} \tilde{f} \to 0.$$

Thus, it suffices to show that $\ker \tilde{f} \cong 0$. To see this consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^{-1}(Y') & \longrightarrow & X & \longrightarrow & X/f^{-1}(Y') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f & & \downarrow \tilde{f} & & \\ 0 & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & X'/Y' & \longrightarrow & 0 \end{array}$$

which has exact rows by definition and the left most square is a pullback. It follows from the pasting laws of pullbacks that

$$\ker \left(f^{-1}(Y') \to Y'\right) \cong \ker f.$$

Hence, the snake lemma implies that

$$\ker \tilde{f} \cong 0.$$

Proving the claim. □

3. The category of Q-shaped modules

Recall that a ring k, may be considered an Ab-enriched category with one object and a (left) k-module M may be considered an additive functor

$$k \xrightarrow{M} \mathrm{Ab}.$$

Suppose, A is an k-algebra, with k a commutative ring. In this situation a (left) A-module in k-modules may be considered an k-linear functor

$$A \to {}_k\mathrm{Mod}.$$

One might study two different things at this point. The first possibility is to study what happens if we replace A by another ${}_k\mathrm{Mod}$-enriched category Q or and the second possibility is to study what happens if we replace the category of k-modules, by another k-linear category. In this section we will study what happens if one does both at the same time. The above analogy is the basis for the next definition.

Definition 3.1. Let k be a commutative ring and Q be a ${}_k\mathrm{Mod}$-enriched and $\mathcal{A}$ a k-linear category. The category of *left Q-shaped modules in $\mathcal{A}$* is the category

$${}_{Q,\mathcal{A}}\mathrm{Mod} := \mathrm{Fun}^k(Q, \mathcal{A})$$

of k-linear functors from Q to $\mathcal{A}$.

Similarly the category of *right Q-shaped modules in $\mathcal{A}$* is the category

$$\mathrm{Mod}_{Q,\mathcal{A}} := \mathrm{Fun}^k(Q^{\mathrm{op}}, \mathcal{A}).$$

We will typically suppress the 'left' in 'left Q-shaped module'.

Remark 3.2. Note that if $\mathcal{A}$ is a k-linear category, then ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is k-linear. This follows as $\mathrm{Fun}(Q, \mathcal{A})$ is k-linear and $\mathrm{Fun}^k(Q, \mathcal{A})$ is a full subcategory of $\mathrm{Fun}(Q, \mathcal{A})$, closed under direct sums, kernels and cokernels. In particular, we may consider $\mathcal{A} \mapsto {}_{Q,\mathcal{A}}\mathrm{Mod}$ an endofunctor

$${}_{Q,-}\mathrm{Mod}\colon \mathrm{Lin}_k \to \mathrm{Lin}_k$$

on the category of k-linear categories. With the action on k-linear functors given by post composition.

Proposition 3.3. *Let $\mathcal{A}$ and $\mathcal{C}$ be k-linear categories and*

$$\mathcal{A} \underset{r}{\overset{l}{\rightleftarrows}} \mathcal{C}$$

be an adjunction. In this situation there exists an adjunction

$${}_{Q,\mathcal{A}}\mathrm{Mod} \underset{r_*}{\overset{l_*}{\rightleftarrows}} {}_{Q,\mathcal{C}}\mathrm{Mod}$$

with l_* given by post composing with l and r_* given by post composing with r.

Proof. The existence of l_* and r_* follow from the fact that

$$\mathcal{A} \mapsto {}_{Q,\mathcal{A}}\mathrm{Mod} = \mathrm{Fun}^k(Q, \mathcal{A})$$

is a functor. So it suffices to show that this is an adjunction. Suppose we have $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$ and $Y \in {}_{Q,\mathcal{C}}\mathrm{Mod}$, in this situation we have

$$\begin{aligned}\mathrm{Hom}_{Q,\mathcal{C}}(l_*X, Y) &\cong \int_{q\in Q} \mathrm{Hom}_{\mathcal{C}}(lX(q), Y(q)) \\ &\cong \int_{q\in Q} \mathrm{Hom}_{\mathcal{A}}(X(q), rY(q)) \\ &\cong \mathrm{Hom}_{Q,\mathcal{A}}(X, r_*Y).\end{aligned}$$

Where the first and third isomorphism is [Kel82, Eq. 2.10], the 2nd uses the adjunction $l \dashv r$ and the fact that l_* and r_* are defined pointwise. □

Proposition 3.4. *Suppose Q is a small ${}_k\mathrm{Mod}$-enriched category and $\mathcal{A}$ is a Grothendieck k-linear category. In this situation, if $\mathcal{A}$ has enough projectives, then ${}_{Q,\mathcal{A}}\mathrm{Mod}$ has enough projectives. Similarly, if $\mathcal{A}$ has enough injectives, then ${}_{Q,\mathcal{A}}\mathrm{Mod}$ has enough injectives.*

Proof. We prove the claim for projectives, as the proof that ${}_{Q,\mathcal{A}}\mathrm{Mod}$ has enough injectives is dual.

Let $i^*\colon {}_{Q,\mathcal{A}}\mathrm{Mod} \to \prod_{q\in Q}\mathcal{A}$ be given by precomposition with the inclusion $i\colon \mathrm{Ob}(Q)\to Q$, this admits a left adjoint given by sending $X\colon \mathrm{Ob}(\mathcal{A}) \to \prod_{q\in Q}\mathcal{A}$ to $\mathrm{Lan}_i X$. We will denote the left adjoint by $i_!$. If $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$, then we may choose an projective object P and a surjection $P \to i^*X$ as $\mathcal{A}$ has enough projectives. We now consider

$$i_!P \to i_!i^*X$$

which is a surjection as $i_!$ is a left adjoint and hence right exact. Now consider the counit of the $i_! \dashv i^*$ adjunction, $i_!i^*X \to X$. We claim this is a surjection. To see this note that i^* is faithful. Hence, the map

$$\mathrm{Hom}_{Q,\mathcal{A}}(X,Y) \to \mathrm{Hom}_{Q,\mathcal{A}}(i_!i^*X,Y)$$

is injective for all Y. It follows that $i_!i^*X \to X$ is an epimorphism and thus an surjection as ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is k-linear. This implies that the composite

$$i_!P \to X$$

is a surjection. Finally, $i_!$ preserves projectives as i^* is exact, so $i_!P$ is projective in ${}_{Q,\mathcal{A}}\mathrm{Mod}$. $\square$

One of the goals for this thesis is to construct (co)homology functors which measures weak equivalences. In order to do so, we will now define the functor tensor product and functor mapping object.

Definition 3.5. Let Q be a small ${}_k\mathrm{Mod}$-enriched category and $\mathcal{A}$ a Grothendieck k-linear category. Let $A \in \mathrm{Mod}_Q$ and $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$. In this situation the *functor tensor product* of A and X, is given by the coend

$$A \otimes_Q X \coloneqq \int^{q\in Q} A(q)\otimes X(q).$$

Similarly, if $B \in {}_Q\mathrm{Mod}$ is a Q-shaped module, the *functor mapping object* of B and X is given by the end

$$\mathrm{map}_Q(B,X) \coloneqq \int_{q\in Q} X(q)^{B(q)}.$$

Remark 3.6. Note that if $\mathcal{A} = {}_k\mathrm{Mod}$, then the tensor is given by the tensor product of k-modules and similarly the cotensor is given by the k-module of k-linear maps. In particular we get an isomorphism

$$\mathrm{map}_Q(B,X) = \int_{q\in Q} X(q)^{B(q)} \cong \int_{q\in Q} \mathrm{Hom}_k(B(q),X(q)) \cong \mathrm{Hom}_Q(B,X).$$

Where the final isomorphism is [Kel82, Eq. 2.10].

Proposition 3.7. *Let $\mathcal{A}$ be a Grothendieck k-linear category. In this situation we have that for all $X \in {}_Q\mathrm{Mod}$ and $Y \in \mathrm{Mod}_Q$ there are adjunctions*

$$\mathcal{A} \underset{\mathrm{map}_Q(X,-)}{\overset{X\otimes -}{\rightleftarrows}} {}_{Q,\mathcal{A}}\mathrm{Mod} \qquad\qquad {}_{Q,\mathcal{A}}\mathrm{Mod} \underset{(-)^Y}{\overset{Y\otimes_Q -}{\rightleftarrows}} \mathcal{A}.$$

Proof. We only prove that the first pair form an adjunction, the other can be proven analogously.

Let $M \in \mathcal{A}$, then we have

$$\begin{aligned}\operatorname{Hom}_{Q,\mathcal{A}}(X \otimes M, N) &\cong \int_{q\in Q} \operatorname{Hom}_{\mathcal{A}}(X(q) \otimes M, N(q)) \\ &\cong \int_{q\in Q} \operatorname{Hom}_{\mathcal{A}}(M, N(q)^{X(q)}) \\ &\cong \operatorname{Hom}_{\mathcal{A}}\left(M, \int_{q\in Q} N(q)^{X(q)}\right) \\ &\cong \operatorname{Hom}_{\mathcal{A}}(M, \operatorname{map}_Q(X, N)).\end{aligned}$$

Here we use the universal property of tensors and cotensors, and the fact that $\operatorname{Hom}_{\mathcal{A}}(M, -)$ preserves ends. □

Remark 3.8. Note that the above implies that $Y \otimes_Q -$ preserves colimits, but in fact more is true. The functor tensor product preserves colimits seperately in each variable. This is easy to see using [Kel82, Eq. 2.10] and the fact that the tensor product defined in Remark 2.6 preserves colimits separately in each variable, which can easily been seen using the universal property of tensors.

Similarly, the functor mapping object preserves limits in each variable separately.

Lemma 3.9. *If $\mathcal{A}$ is a Grothendieck k-linear category. Then for all $X \in {}_Q\mathrm{Mod}$, $Y \in {}_{Q,\mathcal{A}}\mathrm{Mod}$ and $M \in {}_k\mathrm{Mod}$, we have that*

$$\operatorname{map}_Q(X^M, Y) \cong \operatorname{map}_Q(X, Y)^M.$$

If $X' \in \mathrm{Mod}_Q$ we have that

$$(M \otimes X') \otimes_Q Y \cong M \otimes (X' \otimes_Q Y).$$

Proof. We prove that $\operatorname{map}_Q(X^M, Y) \cong \operatorname{map}_Q(X, Y)^M$. The other claim can be proven analogously.

For all $Z \in \mathcal{A}$ we have that

$$\begin{aligned}\operatorname{Hom}_{\mathcal{A}}(Z, \operatorname{map}_Q(X^M, Y)) &\cong \operatorname{Hom}_{\mathcal{A}}\left(Z, \int_{p\in Q} Y(p)^{X(p)^M}\right) \\ &\cong \int_{p\in Q} \operatorname{Hom}_{\mathcal{A}}(Z, Y(p)^{X(p)^M}) \\ &\cong \int_{p\in Q} \operatorname{Hom}_k(M, \operatorname{Hom}_{\mathcal{A}}(Z, Y(p)^{X(p)})) \\ &\cong \operatorname{Hom}_k\left(M, \operatorname{Hom}_{\mathcal{A}}\left(Z, \int_{p\in Q} Y(p)^{X(p)}\right)\right) \\ &\cong \operatorname{Hom}_{\mathcal{A}}\left(Z, \operatorname{map}_Q(X, Y)^M\right).\end{aligned}$$

Thus, the claim follows from the Yoneda lemma. □

Proposition 3.10. *Let $\mathcal{A}$ be a Grothendieck k-linear category. If Q is any small ${}_k\mathrm{Mod}$-enriched category, then for any $q \in Q$ the functor*

$${}_{Q,\mathcal{A}}\mathrm{Mod} \xrightarrow{\mathrm{ev}_q} \mathcal{A},$$

given by evaluating at q, admits both adjoints. It follows that ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is a Grothendieck k-linear category.

Proof. Consider the composite

$$Q \times \mathcal{A} \xrightarrow{Q(q,-)\times\mathcal{A}} {}_k\mathrm{Mod} \times \mathcal{A} \xrightarrow{-\otimes-} \mathcal{A}.$$

We claim that the adjunct

$$\mathcal{A} \longrightarrow \mathrm{Fun}^k(Q, \mathcal{A}) = {}_{Q,\mathcal{A}}\mathrm{Mod}$$

is left adjoint to ev_q. Let us denote it by F_q.

$$\begin{aligned}\mathrm{Hom}_{Q,\mathcal{A}}(F_q(M),N) &= \mathrm{Hom}_{Q,\mathcal{A}}(Q(q,-)\otimes M,N)\\ &\cong \int_{p\in Q}\mathrm{Hom}_{\mathcal{A}}(Q(q,p)\otimes M,N(p))\\ &\cong \int_{p\in Q}\mathrm{Hom}_{k}(Q(q,p),\mathrm{Hom}_{\mathcal{A}}(M,N(p)))\\ &\cong \mathrm{Hom}_{Q}(Q(q,-),\mathrm{Hom}_{\mathcal{A}}(M,N))\\ &\cong \mathrm{Hom}_{\mathcal{A}}(M,N(q)).\end{aligned}$$

Where the first and third isomorphism is [Kel82, Eq. 2.10], the 2nd is universal property of the tensor in $\mathcal{A}$ and the final isomorphism is the Yoneda lemma. The claim that ev_q admits a right adjoint is done similarly, by cotensoring.[3]

To see that ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is Grothendieck, note that it is cocomplete as it is a functor category and $\mathcal{A}$ is cocomplete. Like wise, filtered colimits are exact, as any (co)limit is computed pointwise and filtered colimits are exact in $\mathcal{A}$. Finally, to see that there is a generator note that since ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is cocomplete it suffices to see that there is a set of generators. If $A\in\mathcal{A}$ is a generator, then we claim that $(F_q(A))_{q\in Q}$ generate ${}_{Q,\mathcal{A}}\mathrm{Mod}$. Suppose $f\colon X\to Y$ is a map in ${}_{Q,\mathcal{A}}\mathrm{Mod}$ and

$$\mathrm{Hom}_{Q,\mathcal{A}}(F_q(A),X)\xrightarrow{f_*}\mathrm{Hom}_{Q,\mathcal{A}}(F_q(A),Y)$$

is 0 for all $q\in Q$. Using the adjunction we see that $f(q)_*=0$ for all Q. Now A is a generator of $\mathcal{A}$, so $f(q)=0$ for all $q\in Q$, thus $f=0$. So $(F_q(A))_{q\in Q}$ generate ${}_{Q,\mathcal{A}}\mathrm{Mod}$. □

Notation. If $\mathcal{A}$ and Q are as above, then from now on, for $q\in Q$ we will denote $\mathrm{ev}_q := E_q$. We will also denote its left adjoint by F_q and right adjoint by G_q.

Remark 3.11. In the proof of Proposition 3.10 we explicitly show that if $G\in\mathcal{A}$ is a generator, then $(F_p(A))_{p\in Q}$ are generators of ${}_{Q,\mathcal{A}}\mathrm{Mod}$. This will become very useful in the later parts of this thesis.

Corollary 3.12. *For E_q, F_q, G_q defined as above, the following hold:*

(1) The functor F_q is exact if $Q(q,p)$ is projective for all $p\in Q$.
(2) The functor G_p is exact if $Q(q,p)$ is projective for all $q\in Q$.

Proof. The claims (1) and (2) are easily seen from the definitions and Lemma 2.7. □

Definition 3.13. Let $A\in\mathrm{Mod}_Q$ be as in Definition 3.5. We define the i'th Q-shaped Tor *functor* based at A to be the i'th left derived functor of the functor of $A\otimes_Q -\colon {}_{Q,\mathcal{A}}\mathrm{Mod}\to\mathcal{A}$. That is

$$\mathrm{Tor}_i^Q(A,-) := \mathbb{L}_i(A\otimes_Q -).$$

The goal for the rest of this section to show that the A based Tor functor is balanced. This is of course expected and the proof is similar to the case of the classical balancing of Tor, in fact it even recovers the classical case when $Q=*$ and $\mathcal{A}={}_k\mathrm{Mod}$. We will also prove the analogous result for $\mathrm{map}_Q(-,-)$.

Definition 3.14. If Q is a ${}_k\mathrm{Mod}$-enriched category. Then a right Q-shaped k-module $X\in\mathrm{Mod}_Q$ is *right flat* if $X\otimes_Q -$ is exact.

Lemma 3.15. *If Q is a small ${}_k\mathrm{Mod}$-enriched category, then any projective P in Mod_Q is right flat.*

Proof. Note that by Proposition 3.10 Mod_Q is generated by the set of objects $(Q(-,q))_{q\in Q}$ projective objects in particular we can reduce to the claim for objects of the form $Q(-,q)$. This is because any other projective is a direct summand of $\oplus_{i\in I}Q(-,q_i)$ for some index set I and

[3]The fact that ev_q admit both adjoints is actually a formal consequence of the fact that it is given precomposition with the functor $q\colon\Delta^0\to Q$ taking $*\in\Delta^0$ to q which admits both adjoints by sending a functor to its left/right kan extension along q.

direct sums are exact in Mod_Q at Mod_Q is Grothendieck. Now by the coYoneda lemma we have that

$$Q(-,q) \otimes_Q X \cong \int^{p\in Q} Q(p,q) \otimes X(p) \cong X(q).$$

So $Q(-,q) \otimes_Q - \cong E_q$, which is exact by Proposition 3.10. □

Lemma 3.16. *If Q is a small ${}_k\mathrm{Mod}$-enriched category and $\mathcal{A}$ is a Grothendieck k-linear category with enough left flats. Then ${}_{Q,\mathcal{A}}\mathrm{Mod}$ has enough functor left flats.*

Proof. For any left flat $L \in \mathcal{A}$, consider the object $F_q(L)$. We claim that $F_q(L)$ is functor left flat. To see this let $X \in \mathrm{Mod}_Q$, then

$$\begin{aligned} X \otimes_Q F_q(L) &= M \otimes_Q (Q(q,-) \otimes L) \\ &\cong (X \otimes_Q Q(q,-)) \otimes L \\ &\cong X(q) \otimes L. \end{aligned}$$

It follows that $- \otimes_Q F_q(L) \cong (E_q(-) \otimes L)$, which is exact since evaluation is exact and L is left flat. To see that ${}_{Q,\mathcal{A}}\mathrm{Mod}$ has enough functor left flats, we note that F_q is right exact for every $q \in Q$ and ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is generated by objects of the form $F_q(G)$, where G is a generator of $\mathcal{A}$. So if we choose a surjection $L' \to G$ where L' is left flat, then

$$F_q(L) \to F_q(G)$$

is surjective for every q. Hence, ${}_{Q,\mathcal{A}}\mathrm{Mod}$ has enough functor left flats. □

We are now able to prove that the functor tensor product is balanced.

Theorem 3.17. *Let Q be a small ${}_k\mathrm{Mod}$-enriched category and $\mathcal{A}$ be Grothendieck k-linear category with enough projectives and enough left flat objects. In this situation for every $X \in \mathrm{Mod}_Q$ and $Y \in {}_{Q,\mathcal{A}}\mathrm{Mod}$ we have that*

$$\mathrm{Tor}_i^Q(X,Y) \cong \mathbb{L}_i(- \otimes_Q Y)(X).$$

Proof. Let $P_\bullet \to X$ and $K_\bullet \to Y$ be a projective resolution and a left flat resolution respectively. Furthermore, consider $\mathrm{Tot}^\oplus(P_\bullet \otimes_Q K_\bullet)$ the total complex of the bicomplex given by $P_\bullet \otimes_Q K_\bullet$. From classical theory we have a first quadrant convergent spectral sequence induced by the row filtration of the bicomplex. The spectral sequence has E^2-page given by

$$E^2_{pq} = \begin{cases} \mathbb{L}_p(- \otimes_Q Y)(X) & \text{if } q = 0 \\ 0 & \text{else.} \end{cases}$$

The spectral sequence abuts to $H_{p+q}(\mathrm{Tot}^\oplus(P_\bullet \otimes_Q K_\bullet))$. This spectral degenerates on the E^2-page, since the E^2-page is concentrated in a single row. It follows that

$$\mathbb{L}_p(- \otimes_Q Y)(X) \cong H_p(\mathrm{Tot}^\oplus(P_\bullet \otimes_Q K_\bullet))$$

for all p. Similarly, using the column filtration of $\mathrm{Tot}^\oplus(P_\bullet \otimes_Q K_\bullet)$ we obtain convergent first quadrant spectral sequence with E^2-page given by

$$E'^2_{pq} = \begin{cases} \mathbb{L}_p(X \otimes_Q -)(Y) & \text{if } q = 0 \\ 0 & \text{else.} \end{cases}$$

This spectral sequence also abuts to $H_{p+q}(\mathrm{Tot}^\oplus(P_\bullet \otimes_Q K_\bullet))$. Now this spectral sequence degenerates at the E'^2-page for the same reason as the above one, so we get that

$$\mathrm{Tor}_p^Q(X,Y) = \mathbb{L}_p(X \otimes_Q -)(Y) \cong H_p(\mathrm{Tot}^\oplus(P_\bullet \otimes_Q K_\bullet)) \cong \mathbb{L}_p(- \otimes_Q Y)(X).$$

Proving the claim. □

Remark 3.18. The above argument is essentially the same as the proof that $\mathrm{Tor}_i^R(-,-)$ is balanced in an advanced class on homological algebra.

Remark 3.19. The argument above may also be used to prove that the right derived functors of $\mathrm{map}_Q(-,-)$ are balanced. We will not go through this argument, but it is completely analogous to the above argument and the classical argument that $\mathrm{Ext}^*_Q(-,-)$ is balanced.

4. Existence, heredity and completeness of cotorsion pairs

Let $\mathcal{A}$ be a bicomplete abelian category. In this situation Hovey's theorem [Hov02, Thm. 2.2] states that there is a bijection between abelian model structures on $\mathcal{A}$ and compatible pairs of cotorsion pairs.

In this section we leverage Hovey's theorem, in order to construct the cotorsion pairs which will be the base of the Hovey triples which define our so called *projective* and *injective* model structures on any *locally Gorenstein category* with enough projectives.

Definition 4.1. A Grothendieck category $\mathcal{A}$ is *locally Gorenstein* if

(1) every objects admits a finite projective resolution if and only if it admits a finite injective resolution.

(2) The numbers

$$\mathrm{FPD}(\mathcal{A}) \coloneqq \sup\{\mathrm{pd}_{\mathcal{A}} X \mid X \in \mathcal{A} \text{ with } \mathrm{pd}_{\mathcal{A}} X < \infty\}$$

and

$$\mathrm{FID}(\mathcal{A}) \coloneqq \sup\{\mathrm{id}_{\mathcal{A}} X \mid X \in \mathcal{A} \text{ with } \mathrm{id}_{\mathcal{A}} X < \infty\}$$

are finite. These are called the *finite projective dimension* and the *finite injective dimension* of $\mathcal{A}$, respectively.

(3) $\mathcal{A}$ has a generator with finite projective dimension.

In this situation, we let $\mathcal{L}_{\mathcal{A}}$ denote the full subcategory of $\mathcal{A}$ spanned by objects with finite projective, equivalently injective, dimension.

Definition 4.2. Let $\mathcal{A}$ be a locally Gorenstein k-linear category. We say that $\mathcal{A}$ is *locally n-Gorenstein* if $FPD = FID = n$.

We want to study Q-shaped modules in a locally Gorestein category $\mathcal{A}$. The following setup, puts conditions on Q which ensures that ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is again locally Gorenstein.

Setup 4.3. Let Q be a small ${}_k\mathrm{Mod}$ enriched category. We say that Q satisfies Setup 4.3 if

(1) Q satisfies the *hom-finiteness* condition. That is, for every $p, q \in Q$ the k-module $Q(p,q)$ is finitely generated and projective.

(2) Q is *locally bounded*. That is, for all $q \in Q$ the sets

$$N_-(q) \coloneqq \{p \in Q \mid Q(p,q) \neq 0\} \qquad \text{and} \qquad N_+(q) \coloneqq \{p \in Q \mid Q(q,p) \neq 0\}$$

are finite.

(3) There exists a k-linear equivalence $\mathbb{S}\colon Q \to Q$ and a natural isomorphism

$$Q(q,p) \cong \mathrm{Hom}_k(Q(p, \mathbb{S}(q)), k).$$

Such a functor is called a *Serre functor*.

(4) Q satisfies the *retraction property*. That is, for every $q \in Q$ the unit map $k \to Q(q,q)$ given by $x \mapsto x \cdot \mathrm{id}_q$ admits a retract, such that $Q(q,q) = (x \cdot \mathrm{id}_q) \oplus \tau_q$.

Furthermore, we say that Q satisfies Setup 4.3* if Q satisfies (1), (2), (3) above and

(4*) Q satisfies the *strong retraction property*. That is, for all $q \in Q$ the unit map $k \to Q(q,q)$ admits a retract, and there exists a collection of complements $(\tau_q)_{q \in Q}$ such that

- the map $\tau_q \otimes_k \tau_q \to Q(q,q) \otimes_k Q(q,q) \xrightarrow{\circ} Q(q,q)$ takes image in τ_q.
- the map $Q(q,p) \otimes_k Q(p,q) \xrightarrow{\circ} Q(p,p)$ takes value in τ_q for all $p, q \in Q$ with $p \neq q$.

Proposition 4.4. *Let Q be a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3 and $\mathcal{A}$ a Grothendieck k-linear category. In this situation the functor*

$${}_{Q,\mathcal{A}}\mathrm{Mod} \xrightarrow{i^*} \prod_{q \in Q} \mathcal{A}$$

has the strong Gorenstein transfer property of [DSS17, Def. 3.4]. Here i^* is the functor given by precomposition with the inclusion $i\colon \mathrm{Ob}(Q) \to Q$. In particular, if $\mathcal{A}$ is locally Gorenstein, then ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is locally Gorenstein.

The following proof is, as far as I can tell, first given by me in this generality. This however is only in principle as my only actual addition to this argument is realizing that it could be done in this generality. The main idea of the proof and the spirit of the proof is exactly as in [DSS17].

Proof. Note that i^* admits both adjoints, given by sending $M \in \prod_{q\in Q} \mathcal{A}$ to the left/right kan extension along i. We denote the left adjoint respectively right adjoint by $i_!$ and i_* respectively. These have rather explicit descriptions. For $q \in Q$ we have that

$$(i_!M)(q) \cong \bigoplus_{p\in Q} Q(q,p) \otimes M_p$$

and

$$(i_*M)(q) \cong \prod_{p\in Q} M_p^{Q(p,q)}$$

respectively. This can easily be seen using the Yoneda lemma and [Kel82, Eq. 4.24].

Now as $N_-(q)$ is a finite set, we have that

$$\prod_{p\in Q} M_p^{Q(p,q)} \cong \bigoplus_{p\in Q} M_p^{Q(p,q)}.$$

It follows that

$$\begin{aligned}(i_!M)(q) &\cong \bigoplus_{p\in Q} Q(q,p) \otimes M_p \\ &\cong \bigoplus_{p\in Q} \operatorname{Hom}_k(Q(p,\mathbb{S}(q)),k) \otimes M_p \\ &\cong \bigoplus_{p\in Q} M_p^{Q(p,\mathbb{S}(q))} \\ &\cong (i_*M)(\mathbb{S}(q)).\end{aligned}$$

Here the first isomorphism is the pointwise formula for left Kan extensions, the second is induced by the Serre functor, the third follows from the fact that $Q(p,\mathbb{S}(q))$ is finitely generated projective, see Lemma 2.7, and the fourth follows from the finite support of $Q(-,\mathbb{S}(q))$ and the pointwise formula for right Kan extensions.

Finally, it follows from [Lan21, Thm. 2.2.1, Cor 2.2.2] that i^* is conservative, hence faithful. Thus i^* has the strong GT-property from [DSS17, Def. 3.4]. The claim that ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is locally Gorenstein now follows from [DSS17, Prop. 3.6]. □

Corollary 4.5. *In the situation of Proposition 4.4 we have that an object $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$ is Gorenstein projective/injective if and only if i^*X is Gorenstein projective/injective. Furthermore, the functors i_* and $i_!$ preserves both projective and injective objects and restrict to exact functors on the full subcategory of $\prod_{q\in Q}\mathcal{A}$ spanned by objects of finite projective dimension.*

Proof. This follows from Proposition 4.4 and [DSS17, Lem. 3.5 and Prop. 3.7] □

The following theorem establishes the existence of two cotorsion pairs. In section 5 these will be part of two Hovey triples, which then constitute two different model structures of interest on ${}_{Q,\mathcal{A}}\mathrm{Mod}$.

Theorem 4.6 ([EEG08, Thm. 2.25, Lem. 2.26, Prop. 2.28]). *If $\mathcal{A}$ is a locally Gorenstein category with enough projectives, then the following two pairs*

$$(\mathrm{GPrj}_{\mathcal{A}}, \mathcal{L}_{\mathcal{A}}) \qquad \text{and} \qquad (\mathcal{L}_{\mathcal{A}}, \mathrm{GInj}_{\mathcal{A}})$$

are complete and hereditary cotorsion pairs in $\mathcal{A}$.

Proposition 4.7. *If $\mathcal{A}$ is a locally Gorenstein category with enough projectives, then $\mathcal{L}_{\mathcal{A}}$ is a thick subcategory. Furthermore, $\mathrm{GPrj}_{\mathcal{A}} \cap \mathcal{L}_{\mathcal{A}} = \mathrm{Prj}_{\mathcal{A}}$ and $\mathcal{L}_{\mathcal{A}} \cap \mathrm{GInj}_{\mathcal{A}} = \mathrm{Inj}_{\mathcal{A}}$.*

Proof. We prove first that $\mathcal{L}_{\mathcal{A}}$ is thick. Suppose $X \in \mathcal{L}_{\mathcal{A}}$ decomposes as $X \cong Y \oplus Z$. Then we need to show that $Y, Z \in \mathcal{L}_{\mathcal{A}}$. We know X has finite projective dimension, so there exists some $n \geqslant 0$ such that

$$\operatorname{Ext}^i_{\mathcal{A}}(X, A) \cong 0$$

for all $i \geqslant n$ and $A \in \mathcal{A}$. Now recall that

$$\operatorname{Ext}^i_{\mathcal{A}}(X, A) \cong \operatorname{Ext}^i_{\mathcal{A}}(Y, A) \oplus \operatorname{Ext}^i_{\mathcal{A}}(Z, A)$$

for all $i \geqslant 0$ and $A \in \mathcal{A}$, so it follows that $\operatorname{pd}_{\mathcal{A}} Y \leqslant \operatorname{pd}_{\mathcal{A}} X$ and $\operatorname{pd}_{\mathcal{A}} Z \leqslant \operatorname{pd}_{\mathcal{A}} Y$, proving that $\mathcal{L}_{\mathcal{A}}$ is closed under direct summands. We now need to show that for any short exact sequence

$$0 \to X \to X' \to X'' \to 0$$

where two of the three objects are in $\mathcal{L}_{\mathcal{A}}$, using the long exact sequence induced by this on $\operatorname{Ext}^*_{\mathcal{A}}(-, Y)$ we see that if any of two has finite projective dimension, so has the third.

We now prove that $\operatorname{GPrj}_{\mathcal{A}} \cap \mathcal{L}_{\mathcal{A}} = \operatorname{Prj}_{\mathcal{A}}$. If $X \in \operatorname{GPrj}_{\mathcal{A}} \cap \mathcal{L}_{\mathcal{A}}$, then we may choose a surjection $P \to X$ where P is projective. Taking the kernel we get a short exact sequence

$$0 \to Y \to P \to X \to 0$$

where $X \in \mathcal{L}_{\mathcal{A}}$. Likewise, P is projective so it has finite projective dimenions, so it follows since $\mathcal{L}_{\mathcal{A}}$ is thick that Y has finite projective dimension. By assumption we also have that $X \in \operatorname{GPrj}_{\mathcal{A}}$ and thus by Theorem 4.6 we have that

$$\operatorname{Ext}^1_{\mathcal{A}}(X, Y) \cong 0.$$

It follows X is projective as $\operatorname{Prj}_{\mathcal{A}}$ is closed under direct summands. The converse is simple as any projective has finite projective dimension and is Gorenstein orojective.

The argument to see that $\mathcal{L}_{\mathcal{A}} \cap \operatorname{GInj}_{\mathcal{A}} = \operatorname{Inj}_{\mathcal{A}}$ is analogous to the above one. □

5. Projective and injective model structures on ${}_{Q,\mathcal{A}}\mathrm{Mod}$

In this section we apply Hovey's theorem on abelian model structures on the cotorsion pairs in Theorem 4.6, in order to construct "projective" and "injective" model structures on ${}_{Q,\mathcal{A}}\mathrm{Mod}$.

Theorem 5.1. *If Q is a small ${}_k\mathrm{Mod}$-enriched satisfying Setup 4.3 and $\mathcal{A}$ is a locally Gorenstein k-linear category with enough projectives, then*

(1) there exists an abelian model structure on ${}_{Q,\mathcal{A}}\mathrm{Mod}$ where the cofibrant objects are the Gorenstein projective objects, the trivial objects are the objects with finite projective dimension and every object is fibrant.

(2) There exists an abelian model structure on ${}_{Q,\mathcal{A}}\mathrm{Mod}$ where every object is cofibrant, the trivial objects are the objects with finite injective dimension, and the fibrant objects are the Gorenstein injective objects.

Proof. Ad (1): We claim that $(\operatorname{GPrj}_{Q,\mathcal{A}}, \mathcal{L}_{Q,\mathcal{A}}, {}_{Q,\mathcal{A}}\mathrm{Mod})$ is a Hovey triple. Note that,

$$(\operatorname{GPrj}_{Q,\mathcal{A}} \cap \mathcal{L}_{Q,\mathcal{A}}, {}_{Q,\mathcal{A}}\mathrm{Mod}) = (\operatorname{Prj}_{Q,\mathcal{A}}, {}_{Q,\mathcal{A}}\mathrm{Mod})$$

by Proposition 4.7, which is complete as ${}_{Q,\mathcal{A}}\mathrm{Mod}$ has enough projectives by Proposition 3.4. Furthermore, it is clear that

$$(\operatorname{GPrj}_{Q,\mathcal{A}}, \mathcal{L}_{Q,\mathcal{A}} \cap {}_{Q,\mathcal{A}}\mathrm{Mod}) = (\operatorname{GPrj}_{Q,\mathcal{A}}, \mathcal{L}_{Q,\mathcal{A}}),$$

which is a complete cotorsion pair by Theorem 4.6. Finally, note that $\mathcal{L}$ is thick by Proposition 4.7. Thus it follows from Theorem 2.22 that there is a unique abelian model structure on ${}_{Q,\mathcal{A}}\mathrm{Mod}$ with cofibrant objects given by ${}_{Q,\mathcal{A}}\mathrm{GPrj}$, acyclic objects given by $\mathcal{L}$ and every object fibrant.

Ad (2): this is done analogously. □

Remark 5.2. Note that since ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is locally Gorenstein the trivial objects in the two model structures above coincide.

Definition 5.3. If Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3 and $\mathcal{A}$ is locally Gorenstein and has enough projectives. We let the *projective model structure* on ${}_{Q,\mathcal{A}}\mathrm{Mod}$ be the first model structures produced in Theorem 5.1 and the *injective model structure* on ${}_{Q,\mathcal{A}}\mathrm{Mod}$ be the second model structure produced in Theorem 5.1.

Proposition 5.4. *Assume that Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3. In this situation, the model categories*

$$({}_{Q,\mathcal{A}}\mathrm{Mod})_{\mathrm{Proj}} \qquad \textit{and} \qquad ({}_{Q,\mathcal{A}}\mathrm{Mod})_{\mathrm{Inj}}$$

have the same weak equivalences. More precisely, for a map $\phi\colon X \to Y$ in ${}_{Q,\mathcal{A}}\mathrm{Mod}$ the following are equivalent

(1) the map factors as $\phi = \pi\iota$, where ι is monic with $\operatorname{coker}\iota \in \mathcal{L}_{Q,\mathcal{A}}$ and π is epic with $\ker\pi \in \mathcal{L}_{Q,\mathcal{A}}$.
(2) The map ϕ is a weak equivalence in the projective model structure on ${}_{Q,\mathcal{A}}\mathrm{Mod}$.
(3) The map ϕ is a weak equivalence in the injective model structure on ${}_{Q,\mathcal{A}}\mathrm{Mod}$.

Proof. By Hovey's theorem Theorem 2.22 the weak equivalences in abelian model categories are exactly the ones that factor as

$$\phi = \pi\iota$$

with ι a trivial cofibration and π a trivial fibration. It follows that (2) and (3) implies (1) as $\mathrm{Prj}_{Q,\mathcal{A}} \subseteq \mathcal{L}_{Q,\mathcal{A}}$ and $\mathrm{Inj}_{Q,\mathcal{A}} \subseteq \mathcal{L}_{Q,\mathcal{A}}$. Conversely, if $\phi = \pi\iota$ as in (1). Then ι is a weak equivalence (in both the projective and injective model structures) by [Hov02, Lem. 5.8], the same result proves that π is a weak equivalence by duality. □

Definition 5.5. If Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3 and $\mathcal{A}$ is a locally Gorenstein k-linear category with enough projectives, then the *Q-shaped derived category of $\mathcal{A}$* is the homotopy category

$$\mathcal{D}_Q(\mathcal{A}) := \mathrm{Ho}({}_{Q,\mathcal{A}}\mathrm{Mod})$$

of ${}_{Q,\mathcal{A}}\mathrm{Mod}$ with the projective model structure.

Remark 5.6. By the fundamental theorem of model categories we have that

$$\mathrm{Ho}(({}_{Q,\mathcal{A}}\mathrm{Mod})_{\mathrm{Proj}}) \cong {}_{Q,\mathcal{A}}\mathrm{Mod}[\mathcal{W}^{-1}] \cong \mathrm{Ho}(({}_{Q,\mathcal{A}}\mathrm{Mod})_{\mathrm{Inj}})$$

where ${}_{Q,\mathcal{A}}\mathrm{Mod}[\mathcal{W}^{-1}]$ is the Kan localization of ${}_{Q,\mathcal{A}}\mathrm{Mod}$ at the class of maps described in Proposition 5.4. It follows that the definition of $\mathcal{D}_Q(\mathcal{A})$ does depend on the choice of model structure in Definition 5.5.

Theorem 5.7. *If Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3, and $\mathcal{A}$ is a locally Gorenstein category with enough projectives. Then the category $\mathrm{GPrj}_{Q,\mathcal{A}}$, respectively $\mathrm{GInj}_{Q,\mathcal{A}}$ is Frobenius with pro-injective objects given by $\mathrm{Prj}_{Q,\mathcal{A}}$ and $\mathrm{Inj}_{Q,\mathcal{A}}$, respectively. Furthermore, $\mathcal{D}_Q(\mathcal{A})$ admits a triangulated structure, such that the functors*

$$\mathrm{GPrj}_{Q,\mathcal{A}}/_{\mathrm{Prj}_{Q,\mathcal{A}}} \to \mathcal{D}_Q(\mathcal{A}) \qquad \textit{and} \qquad \mathrm{GInj}_{Q,\mathcal{A}}/_{\mathrm{Inj}_{Q,\mathcal{A}}} \to \mathcal{D}_Q(\mathcal{A})$$

are triangle equivalences.

Proof. The cotorsion pairs determining the projective, respectively injective model structure are all hereditary by Theorem 4.6. Hence, the conclusion follows from Theorem 2.25 and Remark 2.26. □

6. Cohomology

In the previous section we constructed two model structures on ${}_{Q,\mathcal{A}}\mathrm{Mod}$, with the same class of weak equivalences. The goal of this section is to construct (co)homology functors, which measure weak equivalences, in the same sense that (co)homology of chain complexes measure quasi-isomorphisms. For this section we will assume Q satisfies Setup 4.3*.

Definition 6.1. For $p, q \in Q$, we let

$$\tau(p,q) \coloneqq \begin{cases} Q(p,q) & p \neq q \\ \tau(p) & p = q. \end{cases}$$

Proposition 6.2. *For $p, q \in Q$ the k-modules $\tau(p,q)$ form an ideal in Q, called the* pseudo-radical *in Q. That is, $\tau(-,-)$ forms a subfunctor of $Q(-,-)$.*

Proof. This follows from condition (4*) in Setup 4.3*. □

Definition 6.3. For $q \in Q$, the *stalk functor* at q is the functor

$$S\langle q\rangle \coloneqq Q(q,-)/\tau(q,-)$$

and the contravariant stalk functor at q is the functor

$$S\{q\} \coloneqq Q(-,q)/\tau(-,q).$$

Lemma 6.4. *If $q \in Q$, then the stalk functor is given by*

$$S\langle q\rangle(p) = \begin{cases} k & \text{if } p = q \\ 0 & \text{else} \end{cases}$$

and the value on morphisms is determined by the direct sum decomposition

$$\tau(q,q) \oplus k \cong Q(q,q).$$

Proof. This follows from the definitions and the fact the cokernels are computed pointwise in ${}_Q\mathrm{Mod}$. □

We are now able to define (co)homology.

Definition 6.5. Suppose Q is a ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3* and $\mathcal{A}$ a locally Gorenstein. For $q \in Q$ we let

$$C_q\colon {}_{Q,\mathcal{A}}\mathrm{Mod} \to \mathcal{A}$$

be the functor given by $C_q(X) = S\{q\} \otimes_Q X$. Where $S\{q\} \otimes_Q -$ is as defined in Definition 3.5.

Similarly we let

$$K_q\colon {}_{Q,\mathcal{A}}\mathrm{Mod} \to \mathcal{A}$$

be the functor defined on objects by $K_q(X) = \mathrm{map}_Q(S\langle q\rangle, X)$, where $\mathrm{map}_Q(S\langle q\rangle, -)$ is as defined in Definition 3.5.

In this situation, the i'th *homology* of X at $q \in Q$ is defined to be

$$H_i^{[q]}(X) \coloneqq \mathbb{L}_i C_q(X)$$

and the i'th *cohomology* at q is defined as

$$H^i_{[q]}(X) \coloneqq \mathbb{R}^i K_q(X).$$

Example 6.6. Suppose $k = \mathbb{Z}$ and consider the category, $\mathrm{Ch}(\mathbb{Z})$, of chain complexes with values Ab. In this situation the stalk functor $S\langle q\rangle$ at q is the chain complex

$$\ldots \to 0 \to 0 \to \mathbb{Z} \to 0 \to 0 \to \ldots$$

with $\mathbb{Z}$ in cohomological degree q and 0 everywhere else. Now for any chain complex $C_\bullet$ we have from Remark 3.6 that $H^i_{[q]}(C_\bullet) = \mathrm{Ext}^i_{\mathrm{Ch}(\mathbb{Z})}(S\langle q\rangle, C_\bullet)$. We want to construct a projective resolution of $S\langle q\rangle$. Consider the disk complex $D^p_{\mathbb{Z}}$ given by

$$\ldots \to 0 \to \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \to 0 \to \ldots$$

where $\mathbb{Z}$ sits in cohomological degree p and $p+1$ with the identity as differential. The disk complex $D^p_{\mathbb{Z}}$ is projective in $\mathrm{Ch}(\mathbb{Z})$ since it is split exact and the entries are projective abelian groups. Now consider the map

$$f_q\colon D^q_{\mathbb{Z}} \to S\langle q\rangle$$

given by the identity on $\mathbb{Z}$ in degree q and 0 everywhere else. This is clearly a surjection and the kernel is given by

$$\ker f_0 \cong S\langle q+1\rangle.$$

Thus if we let $f_{q+i}\colon D^{q+i}_{\mathbb{Z}} \to S\langle q+i\rangle$ be the map given by the identity of $\mathbb{Z}$ in degree $q+i$ and 0 elsewhere we obtain a projective resolution

$$\ldots \to D^{q+2}_{\mathbb{Z}} \xrightarrow{f_{q+2}} D^{q+1}_{\mathbb{Z}} \xrightarrow{f_{q+1}} D^{q}_{\mathbb{Z}} \to S\langle q\rangle \to 0.$$

In particular the cohomology of

$$\ldots \to \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(D^{q+2}_{\mathbb{Z}}, C_\bullet) \to \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(D^{q+1}_{\mathbb{Z}}, C_\bullet) \to \mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(D^{q}_{\mathbb{Z}}, C_\bullet) \to 0$$

is exactly $\mathrm{Ext}^*_{\mathrm{Ch}(\mathbb{Z})}(S\langle q\rangle, C_\bullet)$. Clearly we have that

$$\mathrm{Hom}_{\mathrm{Ch}(\mathbb{Z})}(D^{p}_{\mathbb{Z}}, C_\bullet) \cong C_p.$$

So it follows that

$$\mathrm{Ext}^i_{\mathrm{Ch}(\mathbb{Z})}(S\langle q\rangle, C_\bullet) \cong H^{q+i}(C_\bullet).$$

Proposition 6.7. *For $q \in Q$, let $S_q\colon \mathcal{A} \to {}_{Q,\mathcal{A}}\mathrm{Mod}$ be the functor given by $S_q(M) := S\langle q\rangle \otimes M$. In this situation, there is a triple adjunction*

$$\mathcal{A} \underset{\underset{K_q}{\longleftarrow}}{\overset{\overset{C_q}{\longleftarrow}}{\xrightarrow{S_q}}} {}_{Q,\mathcal{A}}\mathrm{Mod}.$$

Proof. Using the computation in Lemma 6.4 we see that $S_q = S\langle q\rangle \otimes - \cong (-)^{S\{q\}}$. The results now follows from the adjunctions established in Proposition 3.7. □

Our next goal is to give a more explicit construction of C_q and K_q respectively.

Definition 6.8. For all $q \in Q$ we define the sets

$$J_q := \coprod_{p\in Q} \tau(q,p) \qquad \text{and} \qquad I_q := \coprod_{p\in Q} \tau(p,q).$$

Lemma 6.9. *If $q \in Q$, then there are exact sequences*

$$\bigoplus_{f\in J_q} Q(\mathrm{cod} f, -) \xrightarrow{\phi} Q(q,-) \to S\langle q\rangle \to 0$$

and

$$\bigoplus_{f\in I_q} Q(-, \mathrm{dom} f) \xrightarrow{\psi} Q(-,q) \to S\{q\} \to 0$$

where the component of ϕ at $\mathrm{cod} f$ is given by $\phi_{\mathrm{cod} f} = Q(f,-)$ and similarly for ψ.

Proof. We will only show that

$$\bigoplus_{f\in J_q} Q(\mathrm{cod} f, -) \xrightarrow{\phi} Q(q,-) \to S\langle q\rangle \to 0$$

is exact. The other case can be proven analogously. By definition $Q(q,-) \to S\langle q\rangle$ is the cokernel of

$$\tau(q,-) \hookrightarrow Q(q,-).$$

So it suffices to prove that

$$\mathrm{Im}(\phi) = \tau(q,-).$$

To prove this it suffices to prove that $\mathrm{Im}(\phi)(p) = \tau(q,p)$ for all $p \in Q$. This follows as the functor

$${}_Q\mathrm{Mod} \xrightarrow{\prod \mathrm{ev}_p} \prod_{p\in Q} {}_k\mathrm{Mod}$$

is exact and conservative and hence creates finite limits and finite colimits. So we are reduced to proving the k-modules

$$\sum_{f\in J_q} \operatorname{Im} Q(f,p) = \tau(q,p)$$

are the same, for all $p \in Q$.

Suppose $f \in J_q$ and $g \in Q(\operatorname{cod} f, p)$, then $G(f,p)(h) = hf$, which is in $\tau(q,p)$ as $f \in \tau(q, \operatorname{cod} f)$ and τ is an ideal. It follows that $\operatorname{Im}(f,p) \subseteq \tau(q,p)$.

Suppose that $f \in \tau(q,p)$, then $f \in J_q$ so

$$\operatorname{Im} Q(f,p) \subseteq \sum_{f\in J_q} \operatorname{Im} Q(f,p).$$

Hence, it suffices to show that $f \in \operatorname{Im} Q(f,p)$. To this end, we consider the map

$$Q(f,p)\colon Q(p,p) \to Q(q,p)$$

which on id_p has value $Q(f,p)(\mathrm{id}_p) = \mathrm{id}_p f = f$. This completes the proof. □

Proposition 6.10. *If $q \in Q$ and $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$, then there are isomorphisms*

$$K_q(X) \cong \ker\left(X(q) \to \prod_{f\in J_q} X(\operatorname{cod} f)\right)$$

and

$$C_q(X) \cong \operatorname{coker}\left(\bigoplus_{f\in I_q} X(\operatorname{dom} f) \to X(q)\right).$$

natural in X.

Proof. We will give the proof for C_q as the proof for K_q is analogous.

Recall the exact sequence

$$\bigoplus_{f\in I_q} Q(-,\operatorname{dom} f) \xrightarrow{\psi} Q(-,q) \to S\{q\} \to 0$$

from Lemma 6.9 and that for $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$ we have that,

$$- \otimes_Q X \colon \mathrm{Mod}_Q \to \mathcal{A}$$

is right exact by Remark 3.8. It follows that we get an exact sequence

$$\left(\bigoplus_{f\in I_q} Q(-,\operatorname{dom} f)\right) \otimes_Q X \xrightarrow{\psi\otimes X} Q(-,q) \otimes_Q X \to S\{q\} \otimes_Q X \to 0.$$

Now using k-linearity of $- \otimes_Q X$ we see that

$$\left(\bigoplus_{f\in I_q} Q(-,\operatorname{dom} f)\right) \otimes_Q X \cong \bigoplus_{f\in I_q} Q(-,\operatorname{dom} f) \otimes_Q X.$$

It follows from the coYoneda lemma [Kel82, Eq. 4.25] that we have an exact sequence

$$\bigoplus_{f\in I_q} X(\operatorname{dom} f) \to X(q) \to C_q(X) \to 0$$

proving the claim. □

Lemma 6.11. *In the setting of Proposition 6.10, we have that*

(1) As subobjects of $X(q)$ the following isomorphism hold

$$K_q(X) \cong \cap_{f\in J_q} \ker X(f).$$

(2) As quotientobjects of $X(q)$ the follow isomorphism hold

$$C_q(X) \cong X(q) / \left(\sum_{f \in I_q} \operatorname{im} f \right).$$

Proof. We prove (1) as (2) can be proven analogously.

We prove the more general claim: For any bicomplete k-linear category $\mathcal{A}$ and any collection of maps $(f_i \colon X \to X_i)_{i \in I}$ we have that

$$\ker \left(X \to \prod_{i \in I} X_i \right) \cong \bigcap_{i \in I} \ker f_i.$$

To see this consider the category $I^{\triangleright}$ which is obtained from I by adjoining a terminal object. Furthermore, consider the diagrams ΔX and $\phi \colon I^{\triangleright} \to \mathcal{A}$ given by $\phi(i) = X_i$ and $\phi(\infty) = 0$, then there is a canonical natural transformation in $\eta \colon \Delta(X) \to \phi$ with components given by $\eta_i = f_i$ for all $i \in I$ and $\eta_\infty = 0$. Now as limits in functor categories are computed pointwise it follows that $\ker \eta$ is the diagram given by $\ker \eta(i) = \ker f_i$ and $\ker \eta(\infty) = X$ and the inclusions as maps. Now limits commutes with limits so

$$\bigcap_{i \in I} \ker f_i \cong \lim_{I^{\triangleright}} \ker \eta \cong \ker \lim_{I^{\triangleright}} \eta \cong \ker \left(X \to \prod_{i \in I} X_i \right).$$

Here the last isomorphism follows from the fact that $I^{\triangleright}$ is connected so $\lim_{I^{\triangleright}} \Delta(X) \cong X$ and since $\phi(\infty) = 0$ we have, under X, that

$$\lim_{I^{\triangleright}} \phi \cong \prod_{i \in I} X_i.$$

Proving the claim. □

Proposition 6.12. *If the pseudo-radical τ is nilpotent. That is, there exists an $N \in \mathbb{N}$ such that $\tau^N = 0$. Then for $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$, the following are equivalent:*

(1) For all $q \in Q$, it holds that $X(q) = 0$.
(2) For all $q \in Q$, it holds that $K_q(X) = 0$.
(3) For all $q \in Q$, it holds that $C_q(X) = 0$.

Proof. It is clear that (1) implies (2) and (3). We will show that (2) implies (1), as the (3) implies (1) case is analogous.

Suppose for a contradiction that $X \neq 0$. In this situation there exists a $q_1 \in Q$ such that $X(q_1) \neq 0$. We consider the exact sequence

$$0 \to K_{q_1}(X) \to X(q_1) \to \prod_{f \in J_{q_1}} X(\operatorname{cod} f).$$

As $X(q_1) \neq 0$, there exists a map $f_1 \colon q_1 \to q_2 \in J_{q_1}$ in τ such that $X(f_1) \neq 0$. As $X(f_1) \neq 0$ it follows that $X(q_2) \neq 0$. We claim that there exists a map $f_2 \colon q_2 \to q_3$ such that $X(f_2 f_1) \neq 0$.

To see this claim, suppose for a contradiction that $X(g f_2) = 0$ for all $g \in J_{q_2}$. In this situation we have that the composite

$$X(q_1) \xrightarrow{X(f_1)} X(q_2) \xrightarrow{\phi} \prod_{f' \in J_{q_2}} X(\operatorname{cod} f')$$

is zero. Now as $K_{q_2}(X) = 0$, then we have that ϕ is monic. It follows that $\phi X(f_1) = 0$ implies that $X(f_1) = 0$, a contradiction.

Reiterating this process, we find a sequence $q_1 \xrightarrow{f_1} q_2 \xrightarrow{f_2} \dots \xrightarrow{f_N} q_{N+1}$ such that

$$X(f_N \dots f_2 f_1) \neq 0$$

this is a contradiction, as τ is nilpotent. □

The next two results (Proposition 6.13 and Lemma 6.17) is our justification for discussing intersections and preimages in the context of general bicomplete k-linear categories in our preliminary section. Using the results proved in our preliminary section, we are able to mimic the arguments given for the corresponding statements of [HJ21, Prop. 7.20 & Lem. 7.24] in our setting.

Proposition 6.13. *Suppose Q satisfies Setup 4.3* and $\mathcal{G}$ is a full subcategory of $\mathcal{A}$ closed under subobjects and extensions. In this situation, we have that for all $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$ there is a short exact sequence*

$$0 \to \bigoplus_{q\in Q} S_qK_q(X) \to X \to X' \to 0.$$

Furthermore, if $X(q) \in \mathcal{G}$ for $q \in Q$, then $K_q(X) \in \mathcal{G}$ and $X'(q) \in \mathcal{G}$ for all $q \in Q$.

Dually, if $\mathcal{H}$ is a full subcategory closed under extensions and quotient objects, then there is a short exact sequence

$$0 \to X'' \to X \to \prod_{q\in Q} S_qC_q(X) \to 0.$$

Furthermore, if $X(q) \in \mathcal{H}$ for all $q \in Q$, then $C_q(X), X''(q) \in \mathcal{H}$ for all $q \in Q$.

Proof. Consider the counit transformation $S_qK_q \xrightarrow{\phi_q} \mathrm{Id}_{q,\mathcal{A}\mathrm{Mod}}$. This is a monomorphism as

$$S_qK_q(X)(p) = \begin{cases} K_q(X) & \text{if } p = q \\ 0 & \text{else.} \end{cases}$$

and the component at p is given by $K_q(X) \to X(q)$ if $p = q$ or $0 \to X(q)$ else. Both of these are monomorphisms. Evaluating at $p \in Q$ one easily sees that

$$\bigoplus_{q\in Q} S_qK_q \xrightarrow{\phi} \mathrm{Id}_{Q,\mathcal{A}\mathrm{Mod}}$$

is a monomorphism. In particular, we get a short exact sequence

$$0 \to \bigoplus_{q\in Q} S_qK_q \xrightarrow{\phi} \mathrm{Id}_{Q,\mathcal{A}\mathrm{Mod}} \to \mathrm{coker}(\phi) \to 0.$$

The first claim now follows from the fact that

$$\mathrm{ev}_X\colon \mathrm{Fun}^k({}_{Q,\mathcal{A}}\mathrm{Mod}, {}_{Q,\mathcal{A}}\mathrm{Mod}) \to {}_{Q,\mathcal{A}}\mathrm{Mod}$$

is exact for all $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$.

Now since $\mathcal{G}$ is assumed to be closed under subobjects we have that $K_q(X) \in \mathcal{G}$ as $K_q(X) \to X(q)$ is a monomorphism by Proposition 6.10 and $X(q) \in \mathcal{G}$. It remains to show that for $X' := \mathrm{coker}(\phi)(X)$ we have that $X'(q) \in \mathcal{G}$ for all $q \in Q$.

Note that since E_q is exact for every $q \in Q$ there is a short exact sequence

$$0 \to K_q(X) \to X(q) \to X'(q) \to 0$$

in $\mathcal{A}$. In particular,

$$X'(q) \cong X(q)/K_q(X) \cong X(q)/\ker\left(X(q) \to \prod_{f\in J_q} X(\mathrm{cod} f)\right).$$

Now $\tau(q,r)$ is finitely generated for all $r \in Q$. Furthermore, since Q is locally bounded there is only finitely many $r \in Q$ such that $\tau(q,r) \neq 0$. It follows there is a finite subset

$$\{f_1, \ldots, f_m\} \subseteq J_q,$$

such that for any $g \in J_q$ we have that

$$g = k_1 f_{g_1} + \ldots + k_{m'} f_{g_{m'}},$$

where $k_1, \dots, k_{m'} \in k$ and $f_{g_i} \in \tau(q, \mathrm{cod} g)$. We claim that

$$K_q(X) \cong \ker \left(X(q) \to \prod_{i=1}^{m} X(\mathrm{cod} f_i) \right).$$

and they represent the same subobject of $X(q)$. It follows that

$$X'(q) \cong X(q)/K_q(X) \cong X(q)/\ker \left(X(q) \to \prod_{i=1}^{m} X(\mathrm{cod}(f_i)) \right).$$

Given this claim, which we will prove at the end, we can proceed by induction on m. If $m = 0$, then $K_q(X) = 0$, so $X'(q) = X(q)$ which is in $\mathcal{G}$. Suppose now that the claim holds for $m-1$, and consider

$$K_m = \ker \left(X(q) \xrightarrow{X(f_m)} X(\mathrm{cod} f_m) \right)$$

and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_q(X) & \longrightarrow & K_m & \longrightarrow & K_m/K_q(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X(q) & \longrightarrow & X(q) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

with exact rows. Applying the snake lemma we get a short exact sequence

$$0 \to K_m/K_q(X) \to X(q)/K_q(X) \to X(q)/K_m \to 0.$$

By the first isomorphism theorem we have that $X(q)/K_m = X(q)/\ker(X(f_m)) \cong \mathrm{im} X(f_m)$, which is a subobject of $X(\mathrm{cod} f_m) \in \mathcal{G}$. By assumption $\mathcal{G}$ is closed under subobjects so $X(q)/K_m \in \mathcal{G}$. It follows by direct computation and the second isomorphism theorem that

$$\begin{aligned} K_m/K_q(X) &\cong \ker(f_m) / \left(\ker \left(X(q) \to \prod_{i=1}^{m-1} X(f_i) \right) \cap \ker(f_m) \right) \\ &\cong \left(\ker(f_m) + \ker \left(X(q) \to \prod_{i=1}^{m-1} X(\mathrm{cod} f_i) \right) \right) / \ker \left(X(q) \to \prod_{i=1}^{m-1} X(\mathrm{cod} f_i) \right) \end{aligned}$$

which is a subobject of

$$X(q)/\ker \left(X(q) \to \prod_{i=1}^{m-1} X(\mathrm{cod} f_i) \right).$$

This is in $\mathcal{G}$, by the inductive hypothesis. This finishes the proof as $\mathcal{G}$ is closed under extensions.

Finally, to see the claim that

$$K_q(X) \cong \ker \left(X(q) \to \prod_{i=1}^{m} X(\mathrm{cod} f_i) \right)$$

we prove that they both satisfy the same universal property. For convenience we denote the right hand side above by K' and the canonical inclusion by $i \colon K' \to X(q)$. Consider the diagram

$$\begin{array}{ccc} K_q(X) & \longrightarrow & X(q) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \prod_{i=1}^{m} X(\mathrm{cod} f_i). \end{array}$$

This commutes as the map $X(q) \to \prod_{i=1}^{m} X(\mathrm{cod} f_i)$ factors through $X(q) \to \prod_{f \in J_q} X(\mathrm{cod} f)$. Thus, it follows from the universal property of kernels that there is a unique map

$$K_q(X) \to K'$$

making the obvious diagram commute. Dually, the composite

$$K' \to X(q) \to X(\operatorname{cod} f_i)$$

is zero for all $1 \leqslant i \leqslant m$. In particular if for $g \in J_q$ we have that $g = k_1 f_{g_1} + \ldots + k_{m'} f_{g_{m'}}$, then

$$\begin{aligned} gi' &= k_1 f_{g_1} i' + \ldots + k_{m'} f_{g_{m'}} i' \\ &= 0. \end{aligned}$$

So the composite

$$K' \to X(q) \to \prod_{f \in J_q} X(\operatorname{cod} f)$$

is zero. Therefor, we get a unique map $K_q(X) \to K'$, making the obvious diagram commute. In particular, it follows from their respective uniqueness that these maps must me mutually inverse of each other. Furthermore, this gives an isomorphism over $X(q)$ by construction.

The statement for $\mathcal{H}$ is proven analogously. □

Remark 6.14. Note that in the above proof we consider the functor category

$$\operatorname{Fun}^k({}_{Q,\mathcal{A}}\mathrm{Mod}, {}_{Q,\mathcal{A}}\mathrm{Mod}),$$

which a priori is a large category, as ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is essentially small. One way to get around this obstacle is to invoke the use of Grothendieck universes and enlarging said universe to one which ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is a small k-linear category. For this reason we will not spend time on set-theoretical problems of this sort. Alternatively, one can easily construct X' pointwise, and check that the association is functorial.

Construction 6.15. Let $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$ be a Q-shaped module in $\mathcal{A}$.

(1) For all $\ell \geqslant 0$ we define

$$X^\ell := \begin{cases} X & \text{if } \ell = 0 \\ \operatorname{coker}\left(\bigoplus_{q \in Q} S_q K_q(X^{\ell-1}) \to X^{\ell-1}\right) & \text{else.} \end{cases}$$

(2) For all $\ell \geqslant 0$ we define

$$X_\ell := \begin{cases} X & \text{if } \ell = 0 \\ \ker\left(X_{\ell-1} \to \prod_{q \in Q} S_q C_q(X_{\ell-1})\right) & \text{else.} \end{cases}$$

Definition 6.16. For all $\ell \geqslant 0$ let τ^ℓ be the ℓ'th power of the pseudo-radical τ. We let

$$J_q^\ell := \coprod_{r \in Q} \tau(q, r) \qquad \text{and} \qquad I_q^\ell := \coprod_{r \in Q} \tau(r, q).$$

Furthermore, we let

$$K_q^\ell(X) := \ker\left(X(q) \to \prod_{f \in J_q^\ell} X(\operatorname{cod} f)\right) \qquad \text{and} \qquad C_q^l(X) := \operatorname{coker}\left(\bigoplus_{f \in I_q^\ell} X(\operatorname{dom} f) \to X(q)\right).$$

Lemma 6.17. *Let $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$ be given. Using the notation of Construction 6.15 we have that:*

(1) For all $\ell \geqslant 0$ and $q \in Q$, we have that

$$X^\ell(q) \cong \operatorname{coker}(K_q^\ell(X) \to X(q)) \qquad \textit{and} \qquad K_q(X^\ell) \cong \operatorname{coker}(K_q^\ell(X) \to K_q^{\ell+1}(X)).$$

(2) For all $\ell \geqslant 0$ and $q \in Q$, we have that

$$X_\ell(q) \cong \ker(X(q) \to C_q^\ell(X)) \qquad \textit{and} \qquad C_q(X_l) \cong \ker(C_q^{\ell+1}(X) \to C_q^\ell(X)).$$

Proof. We prove (1), as (2) can be proven analogously.

We proceed by induction on ℓ. For $\ell = 0$ we have $X^0(q) = X(q)$ and $K_q^0(X) = 0$ as $\tau^0 = Q(-,-)$, and thus for all $q \in Q$ we have that $\mathrm{id}_q \in J_q^0$. This implies there exists a mono

$$K_q^0(X) \to \ker(X(q) \xrightarrow{\mathrm{id}_q} X(q)) \cong 0.$$

Therefore, we must have $K_q^0(X) \cong 0$. This part follows from the fact that for all subsets $S \subseteq J_q^0$ there is a commutative diagram

$$\begin{array}{ccc} X(q) & \longrightarrow & \prod_{f \in J_q^0} X(\mathrm{cod} f) \\ \big\downarrow{\scriptstyle X(\mathrm{id}_q)} & & \big\downarrow{\scriptstyle p_S} \\ X(q) & \longrightarrow & \prod_{g \in S} X(\mathrm{cod} g) \end{array}$$

such that the induced map on kernels fits into a commutative diagram

$$\begin{array}{ccccc} K_q^0(X) & \longrightarrow & X(q) & \longrightarrow & \prod_{f \in J_q^0} X(\mathrm{cod} f) \\ \big\downarrow & & \big\downarrow{\scriptstyle X(\mathrm{id}_q)} & & \big\downarrow{\scriptstyle p_S} \\ \ker\left(X(q) \to \prod_{g \in S} X(\mathrm{cod} g)\right) & \longrightarrow & X(q) & \longrightarrow & \prod_{g \in S} X(\mathrm{cod} g) \end{array}$$

and thus must be a mono, as $X(\mathrm{id}_q)$ is a mono.

By definition $K_q^1(X) = K_q(X)$, so we see that the case $\ell = 0$ holds. Assume now that the lemma holds for some ℓ. We may consider the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_q^\ell(X) & \longrightarrow & K_q^{\ell+1}(X) & \longrightarrow & \mathrm{coker}(K_q^\ell(X) \to K_q^{\ell+1}(X)) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & X(q) & \longrightarrow & X(q) & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

By induction $\mathrm{coker}(K_q^\ell(X) \to K_q^{\ell+1}(X)) \cong K_q(X^\ell)$. Using the snake lemma we get an short exact sequence

$$0 \to K_q(X^\ell) \to X(q)/K_q^\ell(X) \to X(q)/K_q^{\ell+1}(X) \to 0.$$

Using the induction hypothesis $X^\ell(q) \cong X(q)/K_q^\ell(X)$ and the definition of $X^{\ell+1}$, we see that

$$X^{\ell+1}(q) \cong X(q)/K_q^{\ell+1}(X).$$

Proving the first formula holds for $\ell + 1$. It remains to show that

$$K_q(X^{\ell+1}) \cong \mathrm{coker}(K_q^{\ell+1}(X) \to K_q^{\ell+2}(X)).$$

For a map $X(g) \colon X(q) \to X(p)$ in Q with induced map

$$X(q)/K_q^{\ell+1}(X) \xrightarrow{X^{\ell+1}(g)} X(p)/K_p^{\ell+1}(X)$$

we proved in Proposition 2.37 that

$$\ker X^{\ell+1}(g) \cong X(g)^{-1}(K_p^{\ell+1}(X))/K_q^{\ell+1}(X).$$

Using this fact and Lemma 6.11 we see that:

$$K_q(X^{\ell+1}) \cong \bigcap_{g \in J_q} \ker X^{\ell+1}(g) \cong \left(\bigcap_{g \in J_q} X(g)^{-1}(K_{\mathrm{cod} g}^{\ell+1}(X)) \right) / K_q^{\ell+1}(X).$$

Therefore, it suffices to prove that $\cap_{g\in J_q} X(g)^{-1}(K^{\ell+1}_{\mathrm{cod}g}(X))$ is isomorphic to $K^{\ell+2}_q(X)$. We compute,

$$\begin{aligned}\bigcap_{g\in J_q} X(g)^{-1}(K^{\ell+1}_{\mathrm{cod}g}(X)) &\cong \bigcap_{g\in J_q} X(g)^{-1}\left(\bigcap_{h\in J^{\ell+1}_{\mathrm{cod}g}} \ker X(h)\right)\\ &\cong \bigcap_{g\in J_q}\bigcap_{h\in J^{\ell+1}_{\mathrm{cod}g}} X(g)^{-1}(\ker X(h))\\ &\cong \bigcap_{g\in J_q}\bigcap_{h\in J^{\ell+1}_{\mathrm{cod}g}} \ker X(hg).\end{aligned}$$

Here the first isomorphism follows from Lemma 6.11, the second follows is proven in Proposition 2.36 and the third follows from the pasting laws of pullbacks. To finish the proof we note now that for $h \in J^{\ell+1}_{\mathrm{cod}g}$ and $g \in J_q$, so it follows that $hg \in J^{\ell+2}_q$, so we get a unique map

$$\bigcap_{g\in J_q}\bigcap_{h\in J^{\ell+1}_{\mathrm{cod}g}} \ker X(hg) \to \bigcap_{f\in J^{\ell+2}_q} \ker X(f)$$

making the obvious diagram commute. Likewise, for any $f \in J^{\ell+2}_q$ we have that f is a k-linear combination of maps

$$q \xrightarrow{g_1} p_1 \xrightarrow{g_2} p_2 \to \ldots \to p_{\ell+1} \xrightarrow{g_{\ell+2}} r$$

with $g_i \in \tau$. It follows that $g_1 \in J_q$ and $h = g_2 \ldots g_{\ell+2} \in J^{\ell+1}_{p_1} = J^{\ell+2}_{\mathrm{cod}g_1}$, so we have that f is a k-linear combination of maps hg where $g \in J_q$ and $h \in J^{\ell+1}_q$, so we get a unique map

$$\bigcap_{f\in J^{\ell+2}_q} \ker X(f) \to \bigcap_{g\in J_q}\bigcap_{h\in J^{\ell+1}_{\mathrm{cod}g}} \ker X(hg)$$

making the obvious diagram commute. It is easy to see that these are mutually inverse of each other. It follows that

$$\bigcap_{g\in J_q}\bigcap_{h\in J^{\ell+1}_{\mathrm{cod}g}} \ker X(hg) \cong \bigcap_{f\in J^{\ell+2}_q} \ker X(f) \cong K^{\ell+2}_q(X).$$

Proving the claim. □

Theorem 6.18. *Assume that the following hold:*

- *The pseudo-radical τ is nilpotent.*
- *$\mathcal{A}$ is a k-linear, locally 1-Gorenstein category with enough projectives.*

In this situation it holds for every object $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$ the following are equivalent:

(1) We have that X belongs to $\mathcal{L}_{Q,\mathcal{A}}$.
(2) For all $G \in \mathrm{GPrj}_{\mathcal{A}}$, $i > 0$ and $q \in Q$, we have that

$$\mathrm{Ext}^i_{Q,\mathcal{A}}(S_q(G), X) \cong 0.$$

(3) For all $G \in \mathrm{GPrj}_{\mathcal{A}}$ and $q \in Q$, we have that

$$\mathrm{Ext}^1_{Q,\mathcal{A}}(S_q(G), X) \cong 0.$$

Furthermore, the following are equivalent

(1) we have that X belongs to $\mathcal{L}_{Q,\mathcal{A}}$.
(2) For all $H \in \mathrm{GInj}_{\mathcal{A}}$, $i > 0$ and $q \in Q$, we have that

$$\mathrm{Ext}^i_{Q,\mathcal{A}}(X, S_q(H)) \cong 0.$$

(3) For all $H \in \mathrm{GInj}_{\mathcal{A}}$ and $q \in Q$ we have that

$$\mathrm{Ext}^1_{Q,\mathcal{A}}(X, S_q(H)) \cong 0.$$

Proof. We will only proof the claim for the projective model structure, as the claim for the injective model structure is analogous.

For (1) implies (2): we note that $S_q(G)$ is Gorenstein projective in ${}_{Q,\mathcal{A}}\mathrm{Mod}$. This follows from Corollary 4.5, as

$$S_q(G)(p) \cong \begin{cases} G & \text{if } p = q \\ 0 & \text{else.} \end{cases}$$

both of which are Gorenstein projective in $\mathcal{A}$. So

$$\mathrm{Ext}^i_{Q,\mathcal{A}}(S_q(G), X) \cong 0$$

since $(\mathrm{GPrj}_{Q,\mathcal{A}}, \mathcal{L}_{Q,\mathcal{A}})$ is a hereditary cotorsion pair.

For (2) implies (3): this is trivial.

For (3) implies (1): we want to prove that for the full subcategory, S, spanned by objects of the form $S_q(G)$, where $q \in Q$ and $G \in \mathrm{GPrj}_{\mathcal{A}}$ we have that $S^\perp$ is contained in $\mathcal{L}_{Q,\mathcal{A}}$. Suppose $Y \in \mathrm{GPrj}_{\mathcal{A}}$. Now consider the sequence of short exact sequences

$$0 \longrightarrow \bigoplus_{q\in Q} S_q K_q(Y) \longrightarrow Y \longrightarrow Y^1 \longrightarrow 0$$

$$0 \longrightarrow \bigoplus_{q\in Q} S_q K_q(Y^1) \longrightarrow Y^1 \longrightarrow Y^2 \longrightarrow 0$$

$$\vdots$$

By assumption τ is nilpotent, so there exists an $N > 0$ such that $\tau^N = 0$. Now this implies that for all $q \in Q$ and $\ell > N$ we have that $J^\ell_q = \{0\}$, it follows that $K^\ell_q(Y^N) = Y(q)$. Now we computed in Lemma 6.17 that

$$K_q(Y^\ell) = \mathrm{coker}(K^\ell_q(Y) \to K^{\ell+1}_q(Y))$$

so $K_q(Y^\ell) \cong \mathrm{coker}(Y(q) \xrightarrow{\mathrm{id}} Y(q)) \cong 0$ for every $q \in Q$. It follows from Proposition 6.12 that $Y(q) = 0$ for all $q \in Q$. It follows that

$$\bigoplus_{q\in Q} S_q K_q(Y^{N-1}) \cong Y^{N-1}$$

This gives us a short exact sequence

$$0 \to \bigoplus_{q\in Q} S_q K_q(Y^{N-2}) \to Y^{N-2} \to \bigoplus_{q\in Q} S_q K_q(Y^{N-1}) \to 0.$$

Note that as ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is locally 1-Gorenstein the full subcategory $\mathcal{G} = \mathrm{GPrj}_{Q,\mathcal{A}}$ is closed under subobjects. It follows from Proposition 6.13 that for all $Y \in {}_{Q,\mathcal{A}}\mathrm{Mod}$ with $Y(q) \in \mathcal{G}$ for all $q \in Q$, we have that $K_q(Y^\ell), Y^\ell(q) \in \mathcal{G}$ for all $\ell > 0$ and $q \in Q$. So suppose $X \in S^\perp$. Then using the long exact sequence in right derived functors we get a long exact sequence

$$0 \to \mathrm{Hom}_{Q,\mathcal{A}}\left(\bigoplus_{q\in Q} S_q K_q(Y^{N-2}), X\right) \to \mathrm{Hom}_{Q,\mathcal{A}}\left(Y^{N-2}, X\right) \to \mathrm{Hom}_{Q,\mathcal{A}}\left(\bigoplus_{q\in Q} S_q K_Q(Y^{N-1}), X\right)$$
$$\to \mathrm{Ext}^1_{Q,\mathcal{A}}\left(\bigoplus_{q\in Q} S_q K_q(Y^{N-2}), X\right) \to \mathrm{Ext}^1_{Q,\mathcal{A}}(Y^{N-2}, X) \to \mathrm{Ext}^1_{Q,\mathcal{A}}\left(\bigoplus_{q\in Q} S_q K_q(Y^{N-1}), X\right) \to \dots$$

It follows that $\mathrm{Ext}^1_{Q,\mathcal{A}}(Y^{N-2}, X) \cong 0$. This holds since for all $i \geqslant 0$ and $q \in Q$ we have that

$$\mathrm{Ext}^1_{Q,\mathcal{A}}\left(\bigoplus_{q\in Q} S_q K_q(Y^i), X\right) \cong \prod_{q\in Q} \mathrm{Ext}^1_{Q,\mathcal{A}}(S_q K_q(Y^i), X) \cong 0$$

since $X \in S$ and $K_q(Y^i) \in \mathcal{G}$ for all $i \geqslant 0$ and $q \in Q$. Working our way down the sequence of short exact sequence we see that

$$\mathrm{Ext}^1_{Q,\mathcal{A}}(Y, X) \cong 0.$$

It now follows, since Y was chosen arbitrarily, that $X \in \mathcal{L}_{Q,\mathcal{A}}$ since $(\mathrm{GPrj}_{Q,\mathcal{A}}, \mathcal{L}_{Q,\mathcal{A}})$ is a cotorsion pair. □

Remark 6.19. In the proof above we show that

$$\{S_q(G) \mid q \in Q, G \in \mathrm{GPrj}_{\mathcal{A}}\}^{\perp} \subseteq \mathrm{GPrj}_{Q,\mathcal{A}}^{\perp} = \mathcal{L}_{Q,\mathcal{A}}.$$

However the other inclusion also holds. This follows as

$$\{S_q(G) \mid q \in Q, G \in \mathrm{GPrj}_{\mathcal{A}}\} \subseteq \mathrm{GPrj}_{Q,\mathcal{A}}$$

and the fact that taking the right orthogonal complements reverses inclusions.

We are now able to prove one of the main results of this thesis.

Theorem 6.20. *Assume that the following hold:*

- *The pseudo-radical τ is nilpotent.*
- *$\mathcal{A}$ is a k-linear, locally Gorenstein and hereditary category with enough projectives. Here hereditary means that the global dimension of $\mathcal{A}$ is $\mathrm{glpd}(\mathcal{A}) \leqslant 1$.*

In this situation, for every object $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$ the following are equivalent:

(1) *We have that X belongs to $\mathcal{L}_{Q,\mathcal{A}}$.*
(2) *For every $q \in Q$ and $i > 0$, we have that $H^i_{[q]}(X) \cong 0$.*
(3) *For every $q \in Q$ we have that $H^1_{[q]}(X) \cong 0$.*

Furthermore, the following are equivalent

(1) *We have that X belongs to $\mathcal{L}_{Q,\mathcal{A}}$.*
(2) *For every $q \in Q$ and $i > 0$, we have that $H_i^{[q]}(X) \cong 0$.*
(3) *For every $q \in Q$ we have that $H_1^{[q]}(X) \cong 0$.*

The proof of the statement in this generality is due to me, and while the idea from the original proof is still present in my proof. There is one major difference. In the paper [HJ21] which this thesis is based on, the authors compare

$$H^i_{[q]}(X) \qquad \text{and} \qquad \mathrm{Ext}^i_{Q,\mathcal{A}}(S_q(P), X)$$

with P projective, both of which are k-modules in their setting. However in the setting of this thesis, $H^i_{[q]}(X) \in \mathcal{A}$ and $\mathrm{Ext}^i_{Q,\mathcal{A}}(S_q(P), X) \in {}_k\mathrm{Mod}$. As such, it is absurd to expect to be able compare them, as they do not live in the same category. However, as the proof will show, it turns out that one vanishes if and only if the other does which is sufficient for this statement.

Proof. We treat the projective case first, and then argue the injective at the end. Note first that

$$\mathrm{glGpd}(\mathcal{A}) \leqslant \mathrm{glpd}(\mathcal{A}) \leqslant 1$$

so Theorem 6.18 applies. Furthermore, $\mathrm{glpd}(\mathcal{A}) \leqslant 1$ implies that

$$\mathrm{Prj}_{\mathcal{A}} = \mathrm{GPrj}_{\mathcal{A}},$$

since any Gorenstein projective object is a subobject of a projective object. It follows that the equivalent conditions of Theorem 6.18 are as follows

(1) the Q-shaped module X belongs to $\mathcal{L}_{Q,\mathcal{A}}$.
(2) The k-module $\mathrm{Ext}^i_{Q,\mathcal{A}}(S_q(P), X)$ vanishes for all $P \in \mathrm{Prj}_{\mathcal{A}}$, $q \in Q$ and $i > 0$.
(3) The k-module $\mathrm{Ext}^1_{Q,\mathcal{A}}(S_q(P), X)$ vanishes for all $P \in \mathrm{Prj}_{\mathcal{A}}$ and $q \in Q$.

Now for $P \in \mathrm{Prj}_{\mathcal{A}}$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(P, H^i_{[q]}(X)) &= \mathrm{Hom}_{\mathcal{A}}(P, \mathbb{R}^i K_q(X)) \\ &\cong \mathbb{R}^i \mathrm{Hom}_{\mathcal{A}}(P, K_q(X)) \\ &\cong \mathbb{R}^i \mathrm{Hom}_{Q,\mathcal{A}}(S_q(P), X) \\ &\cong \mathrm{Ext}^i_{Q,\mathcal{A}}(S_q(P), X). \end{aligned}$$

Here the first isomorphism follows as $\operatorname{Hom}_{\mathcal{A}}(P,-)$ is exact, the second follows by adjunction and the third is by definition.

In particular if $H^i_{[q]}(X) \cong 0$, then

$$\operatorname{Ext}^i_{Q,\mathcal{A}}(S_q(P), X) \cong 0.$$

So $X \in \mathcal{L}_{Q,\mathcal{A}}$. The converse also holds. This follows as $\mathcal{A}$ admits a generator G and has enough projectives. To see this claim let $P \in \operatorname{Prj}_{\mathcal{A}}$ and $P \xrightarrow{\phi} G$ be a surjection, the kernel $\ker(\phi)$ is projective as $\mathcal{A}$ is hereditary. Using the long exact sequence in $\operatorname{Ext}^*_{\mathcal{A}}(-, H^i_{[q]}(X))$ we see that

$$\operatorname{Hom}_{\mathcal{A}}(G, H^i_{[q]}(X)) \cong 0.$$

It follows, since G is a generator, that if $\operatorname{Ext}^i_{Q,\mathcal{A}}(S_q(P), X) \cong 0$ for all $i > 0$, $P \in \operatorname{Prj}_{\mathcal{A}}$ and $q \in Q$, then

$$H^i_{[q]}(X) \cong 0.$$

This proof also specializes to the statement for $i = 1$.

Suppose now that $\mathcal{A}$ has enough injectives. Let $I \in \operatorname{Inj}_{\mathcal{A}}$ be injective, then for all $i > 0$ and $q \in Q$ we have

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{A}}(H^{[q]}_i(X), I) &= \operatorname{Hom}_{\mathcal{A}}(\mathbb{L}_i C_q(X), I) \\
&\cong \mathbb{R}_i \operatorname{Hom}_{\mathcal{A}}(C_q(X), I) \\
&\cong \mathbb{R}_i \operatorname{Hom}_{Q,\mathcal{A}}(X, S_q(I)) \\
&\cong \operatorname{Ext}^i_{Q,\mathcal{A}}(X, S_q(I)).
\end{aligned}$$

Where the first isomorphism, follows from the fact that $\operatorname{Hom}_{\mathcal{A}}(-, I)$ is exact, the 2nd follows by adjunction and the third holds by definition. The rest of the proof is analogous. □

We have now shown that a Q-shaped module in $\mathcal{A}$ is weakly equivalent to 0 if and only if it is "acyclic" with respect to (co)homology. The next theorem states that a map is a weak equivalence if and only if it is a "quasi-isomorphism". That is an $H^*_{[q]}(-)$ isomorphism.

Theorem 6.21. *Assume that the following hold:*

- *The pseudo-radical τ is nilpotent.*
- *$\mathcal{A}$ is a k-linear, locally Gorenstein and hereditary category with enough projectives.*

For a map $\phi\colon X \to Y$ of Q-shaped modules in $\mathcal{A}$ the following are equivalent:

(1) The map $\phi\colon X \to Y$ is a weak equivalence.
(2) The map $H^i_{[q]}(\phi)\colon H^i_{[q]}(X) \to H^i_{[q]}(Y)$ is an isomorphism for all $i > 0$ and $q \in Q$.
(3) The map $H^i_{[q]}(\phi)\colon H^i_{[q]}(X) \to H^i_{[q]}(Y)$ is an isomorphism for $i \in \{1, 2\}$ and $q \in Q$.

Furthermore, the following are equivalent

(1) The map $\phi\colon X \to Y$ is a weak equivalence.
(2) The map $H^{[q]}_i(\phi)\colon H^{[q]}_i(X) \to H^{[q]}_i(Y)$ is an isomorphism for all $i > 0$ and $q \in Q$.
(3) The map $H^{[q]}_i(\phi)\colon H^{[q]}_i(X) \to H^{[q]}_i(Y)$ is an isomorphism for $i \in \{1, 2\}$ and $q \in Q$.

Proof. We prove the projective case first.

For (1) implies (2): Suppose $\phi\colon X \to Y$ is a weak equivalence. As ${}_{Q,\mathcal{A}}\operatorname{Mod}$, with the projective model structure, is a model category there exists a factorization

$$\phi = \pi\iota$$

with $\iota\colon X \to Z$ a trivial cofibration and $\pi\colon Z \to Y$ a trivial fibration. In particular, ι is monic with $\operatorname{coker}\iota \in \operatorname{Prj}_{Q,\mathcal{A}}$ by Proposition 4.7. It follows that $\operatorname{coker}\iota \in \mathcal{L}_{Q,\mathcal{A}}$, so $H^i_{[q]}(\operatorname{coker}\iota) \cong 0$ for all $i > 0$ and $q \in Q$, by Theorem 6.20. Now the short exact sequence

$$0 \to X \xrightarrow{\iota} Z \xrightarrow{p} \operatorname{coker}\iota \to 0$$

is split, as $\operatorname{coker}\iota$ is projective. Hence, it follows that

$$0 \to H^i_{[q]}(X) \xrightarrow{H^i_{[q]}(\iota)} H^i_{[q]}(Z) \xrightarrow{H^i_{[q]}(p)} H^i_{[q]}(\operatorname{coker}\iota) \to 0$$

is an exact sequence for all $i > 0$ and $q \in Q$. Taking these two facts together we get that

$$H^i_{[q]}(\iota)\colon H^i_{[q]}(X) \xrightarrow{\cong} H^i_{[q]}(Z)$$

is an isomorphism for all $i > 0$ and $q \in Q$. Similarly, we have that $\pi\colon Z \to Y$ is an epimorphism with $\ker\pi \in \mathcal{L}_{Q,\mathcal{A}}$, it follows from Theorem 6.20 that

$$H^i_{[q]}(\ker\pi) \cong 0$$

for all $i > 0$ and $q \in Q$. Using the long exact sequence for the right derived functors of K_q induced by the short exact sequence

$$0 \to \ker\pi \xrightarrow{i} Z \xrightarrow{\pi} Y \to 0.$$

We get a long exact sequence

$$\ldots \to H^1_{[q]}(\ker\pi) \to H^1_{[q]}(Z) \to H^1_{[q]}(Y) \to H^2_{[q]}(\ker\pi) \to \ldots$$

So the map

$$H^i_{[q]}(\pi)\colon H^i_{[q]}(Z) \xrightarrow{\cong} H^i_{[q]}(Y)$$

is an isomorphism for all $i > 0$ and $q \in Q$. It follows that

$$H^i_{[q]}(\phi) = H^i_{[q]}(\pi\iota) = H^i_{[q]}(\pi)H^i_{[q]}(\iota)$$

is an isomorphism for all $i > 0$ and $q \in Q$.

For (2) implies (3): this is trivial.

For (3) implies (1): Suppose $\phi\colon X \to Y$ is an isomorphism on cohomology for all $q \in Q$ and $i \in \{1, 2\}$. Since ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is a model category with the projective model structure, we may find a factorization

$$\phi = \pi\iota$$

with $\iota\colon X \to Z$ a cofibration and $\pi\colon Z \to Y$ a trivial fibration. By the 2-out-of-3 property for model categories, it suffices to show that ι is a weak equivalence and thus a trivial cofibration. Now since ι is a cofibration and ${}_{Q,\mathcal{A}}\mathrm{Mod}$ is an abelian model category with the projective model structure, it suffices to show that $\operatorname{coker}\iota \in \mathcal{L}_{Q,\mathcal{A}}$.

To see this, we use the long exact sequence for right derived functors of K_q induced by the short exact sequence

$$0 \to X \xrightarrow{\iota} Y \to \operatorname{coker}\iota \to 0$$

to get the following exact sequence:

$$\ldots \to H^1_{[q]}(X) \xrightarrow{\cong} H^1_{[q]}(Z) \to H^1_{[q]}(\operatorname{coker}\iota) \to H^2_{[q]}(X) \xrightarrow{\cong} H^2_{[q]}(Z) \to \ldots$$

Here $H^i_{[q]}(\iota)$ is an isomorphism for $i \in \{1, 2\}$ since π is a weak equivalence, and hence by (1) implies (2) an isomorphism on cohomology for all $i > 0$ and $q \in Q$, we also have that ϕ is an isomorphism on cohomology in degrees 1 and 2 for all $q \in Q$. It follows that $H^1_{[q]}(\operatorname{coker}\iota) \cong 0$, so $\iota\colon X \to Z$ is a trivial cofibration by Theorem 6.20. It follows that $\phi = \pi\iota$ is a weak equivalence, by the 2-out-of-3 property for weak equivalences.

The injective case is proven dually. □

Remark 6.22. The last two theorems begs the idea that cohomology of a Q-shaped module is controlled by what happens in low cohomological degrees. Indeed, we have proven in the case of chain complexes in Example 6.6 that

$$H^i_{[q]}(X) \cong H^{i+q}(X)$$

for all chain complexes X. Here $H^{i+q}(-)$ is the $(i + q)$'th cohomology of chain complex X. Therefor, we believe one should think of cohomology as living in degree 1 (and 2) and changing in q.

We will now work towards proving that the assumption that maps induce isomorphisms in cohomological degree 2 is unnecessary in certain cases, when proving that they are weak equivalences. This explains why the case of chain complexes differs from the results of Theorem 6.21.

Lemma 6.23. *Suppose there exists some $\ell \geqslant 0$ such that $(\tau^\ell/\tau^{\ell+1})(p,q)$ is free for all $p,q \in Q$. In this situation*

(1) For every $p \in Q$ there exists a collection of sets $(U_q)_{q\in Q}$ and an isomorphism

$$(\tau^\ell/\tau^{\ell+1})(p,-) \cong \bigoplus_{q\in Q} S\langle q\rangle^{k[U_q]}$$

of Q-shaped modules in ${}_k\mathrm{Mod}$. Where $k[U_q]$ is the free k-module on U_q.

(2) For every $p \in Q$ there exists a collection of sets $(V_q)_{q\in Q}$ and an isomorphism

$$(\tau^\ell/\tau^{\ell+1})(-,p) \cong \bigoplus_{q\in Q} S\{q\}^{k[V_q]}$$

of Q-shaped modules in ${}_k\mathrm{Mod}$. Where $k[V_q]$ is the free k-module on V_q.

Proof. We prove (1) as (2) can be proven analogously.

We fix $p \in Q$ for the entirety of this proof. For all $q \in Q$ choose a basis U_q of $(\tau^\ell/\tau^{\ell+1})(p,q)$ and for all $x \in U_q$ let ε_x be a lift of x along $\tau^\ell(p,q) \to (\tau^\ell/\tau^{\ell+1})(p,q)$. For all $x \in U_q$, we have natural transformation $\varepsilon_x\colon Q(q,-) \to Q(p,-)$, and $\tau^\ell(-,-)$ is an ideal in Q and $\varepsilon_x \in \tau^\ell(p,q)$ so we get an induced map

$$Q(\varepsilon_x,-)\colon Q(q,-) \to \tau^\ell(p,-)$$

which map the subfunctor $\tau(q,-)$ to the subfunctor $\tau^{\ell+1}(p,-)$. It follows by passing to cokernels that we get a map

$$S\langle q\rangle = Q(q,-)/\tau(q,-) \to (\tau^\ell/\tau^{\ell+1})(p,-).$$

Using the universal property for cotensors we get an induced map

$$\sigma_q\colon S\langle q\rangle^{k[U_q]} \to (\tau^\ell/\tau^{\ell+1})(p,-).$$

Similarly, by the universal property of coproducts we get an induced map

$$\sigma\colon \bigoplus_{q\in Q} S\langle q\rangle^{k[U_q]} \to (\tau^\ell/\tau^{\ell+1})(p,-).$$

We claim that σ is an isomorphism. We check that all components are isomorphisms, that is for all $r \in Q$ we have that

$$k^{k[U_q]} \cong S\langle r\rangle(r)^{k[U_q]} \cong \bigoplus_{q\in Q} S\langle q\rangle^{k[U_q]}(r) \to (\tau^\ell/\tau^{\ell+1})(p,r)$$

is an isomorphism. This is true by construction. Which is what we wanted to prove. □

Remark 6.24. Note that by construction the sets U_q are finite for all $q \in Q$, it follows that $k[U_q]$ is finitely generated of rank $\#U_q$ for all $q \in Q$.

Remark 6.25. In the formula of Lemma 6.23 we cotensor with $k[U_p]$, this is the same as tensoring since U_p is finite, and the dual of a free k-module of finite rank is a free k-module of the same rank.

Theorem 6.26. *If Q is a ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3* such that $\tau^2 = 0$, the ring k is a principal ideal domain and $\mathcal{A}$ is Hereditary category with enough projectives. Then for a map $\phi\colon X \to Y$ in ${}_{Q,\mathcal{A}}\mathrm{Mod}$ the following are equivalent:*

(1) The map $\phi\colon X \to Y$ is a weak equivalence.
(2) The map $H^1_{[q]}(\phi)\colon H^1_{[q]}(X) \to H^1_{[q]}(Y)$ is an isomorphism for all $q \in Q$.
(3) The map $H^{[q]}_1(\phi)\colon H^{[q]}_1(X) \to H^{[q]}_1(Y)$ is an isomorphism for all $q \in Q$.

Proof. For (1) implies (2), by assumption Theorem 6.21 applies, so the conclusion follows.

For (2) implies (1), as k is a PID we have that $Q(q,p)$ is free for all $q,p \in Q$, it follows $\tau(q,p)$ is also free for all $q,p \in Q$. In particular, since

$$(\tau/\tau^2)(q,p) \cong \tau(q,p)$$

for all $q,p \in Q$ by assumption Lemma 6.23 applies. It follows that for all $p \in Q$ there exists a collection of sets $(U_q)_{q\in Q}$ such that

$$\tau(p,-) \cong (\tau/\tau^2)(p,-) \cong \bigoplus_{q\in Q} S\langle q\rangle^{k[U_q]}.$$

It follows that for all from the long exact sequence of right derived functors of $\mathrm{map}_Q(-,X)$ applied to the short exact sequence

$$0 \to \tau(p,-) \to Q(p,-) \to S\langle p\rangle \to 0$$

that

$$H^2_{[q]}(X) = \mathbb{R}^2\mathrm{map}_Q(S\langle p\rangle, X) \cong \mathbb{R}^1\mathrm{map}_Q(\tau(p,-),X)$$

for all $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$. We now compute that

$$\begin{aligned}\mathbb{R}^1\mathrm{map}_Q(\tau(p,-),X) &\cong \mathbb{R}^1\mathrm{map}_Q\left(\bigoplus_{q\in Q} S\langle q\rangle^{k[U_q]}, X\right)\\ &\cong \prod_{q\in Q} \mathbb{R}^1\mathrm{map}_Q(S\langle q\rangle^{k[U_q]}, X)\\ &\cong \prod_{q\in Q} (\mathbb{R}^1\mathrm{map}_Q(S\langle q\rangle, X))^{k[U_p]}\\ &\cong \prod_{q\in Q} H^1_{[q]}(X)^{k[U_q]}.\end{aligned}$$

Where the first isomorphism follows from the above computation. The 2nd follows because the fact that $\mathrm{map}_Q(-,-)$ has balanced derived functors and the third follows from Lemma 3.9 and the fourth is the definition.

Here the isomorphism is natural in X since $\mathrm{map}_Q(-,-)$ is a bifunctor. It follows that $H^2_{[q]}(\phi)\colon H^2_{[q]}(X) \to H^2_{[q]}(Y)$ is an isomorphism for all $q \in Q$, so the conclusion follows from Theorem 6.21. Proving the claim. □

We will end this section by giving a characterization of the projective and injective objects in ${}_{Q,\mathcal{A}}\mathrm{Mod}$. To do this recall that the functor $E_q\colon {}_{Q,\mathcal{A}}\mathrm{Mod} \to \mathcal{A}$ given by evaluating at $q \in Q$ admits a both adjoints given by tensoring pointwise with $Q(q,-)$ and cotensoring with $Q(-,q)$ respectively. We denote the left adjoint by F_q and the right adjoint by G_q.

Lemma 6.27. *Suppose Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3, let $\mathcal{A}$ be a Grothendieck k-linear category, let $M \in \mathcal{A}$ and let $p \in Q$. In this situation we have:*

(1) $H^{[q]}_i(F_p(M)) \cong 0$ for all $i > 0$ and every $q \in Q$.
(2) $H^i_{[q]}(G_p(M)) \cong 0$ for all $i > 0$ and every $q \in Q$.

Proof. We prove (1) as (2) can be proven analogously.

Let $P_\bullet \to S\{q\}$ be a projective resolution of $S\{q\}$ in Mod_Q. By Theorem 3.17 we have that

$$H_i(P_\bullet \otimes_Q F_p(M)) \cong \mathbb{L}_i S\{q\} \otimes_Q F_p(M) = H^{[q]}_i(F_p(M)).$$

Now using the coYoneda lemma and Lemma 3.9 we have that

$$\begin{aligned}P_\bullet \otimes_Q F_p(M) &\cong P_\bullet \otimes_Q (Q(p,-)\otimes M)\\ &\cong (P_\bullet \otimes_Q Q(p,-)) \otimes M\\ &\cong P_\bullet(p) \otimes M.\end{aligned}$$

Now E_p is exact and its right adjoint G_p is exact by Corollary 3.12, so $f\colon P_\bullet(p) \to S\{q\}(p)$ is a projective resolution. Now $S\{q\}(p)$ is either k or 0 by Lemma 6.4 and hence, so f is a homotopy equivalence. It follows that $P_\bullet(p) \otimes M$ is either homotopy equivalent to M or 0. Both of these complexes has no homology in positive degrees. It follows that

$$H_i^{[q]}(F_p(M)) \cong H_i(P_\bullet(p) \otimes M) \cong 0$$

for all $i > 0$ and every $q \in Q$. □

Lemma 6.28. *Let Q be small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3 and let $\mathcal{A}$ be any Grothendieck k-linear category. For every $p, q \in Q$ we have that*

(1) $C_pF_p = \mathrm{Id}_\mathcal{A}$ and $C_pF_q = 0$ if $p \neq q$.
(2) $K_pG_p = \mathrm{Id}_\mathcal{A}$ and $K_pG_q = 0$ if $p \neq q$.

Proof. We prove (1) as (2) can be proven analogously.

Since $- \otimes_Q -$ is a bifunctor we may compute this pointwise. So suppose $M \in \mathcal{A}$. Then we have

$$\begin{aligned} C_pF_q(M) &= S\{p\} \otimes_Q (Q(q,-) \otimes M) \\ &\cong (S\{p\} \otimes_Q Q(q,-)) \otimes M \\ &\cong S\{p\}(q) \otimes M. \end{aligned}$$

Now in Lemma 6.4 we proved that $S\{q\}(p)$ is k if $p = q$ and 0 else. This proves the claim. □

We are now able to prove our characterization of projective and injective objects in ${}_{Q,\mathcal{A}}\mathrm{Mod}$.

Theorem 6.29. *Let Q be a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3* and assume that the pseudo-radical τ is nilpotent.Furthermore, let $\mathcal{A}$ be any Grothendieck k-linear category. In this situation, for every $X \in {}_{Q,\mathcal{A}}\mathrm{Mod}$ we have that:*

(1) The object X is projective if and only if $H_1^{[q]}(X) = 0$ and $C_q(X)$ is projective for every $q \in Q$.
(2) The object X is injective if and only if $H^1_{[q]}(X)$ and $K_q(X)$ is injective for every $q \in Q$.

Proof. We prove (1) as (2) is proven analogously.

Suppose $X \in \mathrm{Prj}_{Q,\mathcal{A}}$ is projective. Then by Proposition 3.10 we may reduce to the case where $X \cong F_q(P)$, where $P \in \mathcal{A}$ is projective and $q \in Q$. In this case we have that $H_1^{[q]}(F_q(P)) \cong 0$ by Lemma 6.27 and

$$C_pF_q(P) \cong \begin{cases} P & \text{if } p = q \\ 0 & \text{else.} \end{cases}$$

by Lemma 6.28, which is projective in both cases.

Suppose now that $C_p(X) \in \mathrm{Prj}_\mathcal{A}$ and $H_1^{[q]}(X) \cong 0$ for every $q \in Q$. Then since $C_q(X)$ is projective the canonical surjection

$$\pi_q\colon X(q) \to C_q(X)$$

admits a section $\iota_q\colon C_q(X) \to X(q)$. We consider the composite

$$\phi_q\colon F_qC_q(X) \xrightarrow{F_q(\iota)} F_qE_q(X) \xrightarrow{\varepsilon_q} X$$

where ε_q is the counit of the adjunction $F_q \vdash E_q$ at X. Using Lemma 6.27 to see that

$$C_q(\varepsilon_q) = \pi_q$$

and $C_q(F_q(\iota_q)) = \iota_q$. It follows that $C_q(\phi_q) = \mathrm{id}_{C_q(X)}$. Using the universal property of coproducts we get a map

$$\bigoplus_{q \in Q} F_qC_q(X) \xrightarrow{\phi} X.$$

We claim that ϕ is an isomorphism. To see that ϕ is surjective consider the exact sequence

$$\bigoplus_{q\in Q} F_q C_q(X) \xrightarrow{\phi} X \to \operatorname{coker}\phi \to 0.$$

We know that C_p is a left adjoint for every p, so the sequence

$$\bigoplus_{q\in Q} C_p F_q C_q(X) \xrightarrow{C_p(\phi)} C_p(X) \to \operatorname{coker} C_p(\phi) \to 0.$$

is exact. Now using the fact that ϕ is the map induced on coproducts by ϕ_p for every $p \in Q$ and Lemma 6.27 we see that $C_p(\phi) = \mathrm{Id}_{C_p(X)}$ on $C_p(X)$, which is surjective so $\operatorname{coker} C_p(\phi) = C_p(\operatorname{coker}\phi) = 0$, which implies that $\operatorname{coker}\phi = 0$ by Proposition 6.12. Note that since $\mathbb{L}_1 C_p(X) \cong H_1^{[p]}(X) \cong 0$ for every $p \in Q$ we have that the sequence

$$0 \to C_p(\ker\phi) \to \bigoplus_{q\in Q} C_p F_q C_q(X) \to C_p(X) \to 0$$

is exact. Now $C_p(\phi) = \mathrm{id}_{C_p(X)}$ is injective so we have that

$$C_p(\ker\phi) \cong 0$$

for all $p \in Q$, so by Proposition 6.12 we have that $\ker\phi = 0$. It follows that

$$\bigoplus_{q\in Q} F_q C_q(X) \xrightarrow{\phi} X$$

is an isomorphism. Now $C_q(X)$ is projective for every $q \in Q$ and F_q preserves projectives, which implies that $F_q C_q(X)$ is projective. Finally, projectives are closed under direct sums, so we get that X is projective. As we wanted to prove. □

7. The Q-shaped derived category of a ring

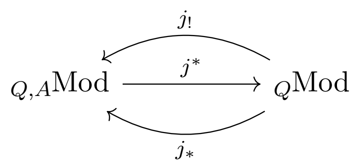
In this section we will focus on $\mathcal{A} = {}_A\mathrm{Mod}$, where A is a k-algebra and k is a Gorenstein ring. We prove that even though ${}_A\mathrm{Mod}$ is not necessarily Gorenstein, we can still construct projective and injective model structures on the category of Q-shaped A-modules ${}_{Q,A}\mathrm{Mod}$, and (co)homology functors, which controls the weak equivalences, similarly to the situation in Theorem 6.21. The results in this section are all from [HJ21]. They are somewhat intertwined with the results of the rest of this thesis in the paper this thesis is based on, so we will make sure to give ample reference to the results as they are stated in [HJ21].

For this section we will assume that k is a commutative Gorenstein ring and A is a k-algebra. Using the fact that ${}_k\mathrm{Mod}$ is locally Gorenstein it follows from Theorem 4.6 that $(\mathrm{GPrj}_{Q,k}, \mathcal{L}_{Q,k})$ and $(\mathcal{L}_{Q,k}, \mathrm{GInj}_{Q,k})$ are cotorsion pairs in ${}_Q\mathrm{Mod}$.

Recall that the restriction of scalars functor

$$j\colon {}_A\mathrm{Mod} \to {}_k\mathrm{Mod}$$

admits both adjoints and is conservative. With the left adjoint given by $-\otimes_k A$ and right adjoint given by $\mathrm{Hom}_k(A,-)$. This implies by Proposition 3.3 that there is a triple adjunction



where j^* is given by pointwise restriction of scalars, $j_!$ is given by pointwise extension of scalars and j_* is given by pointwise coextension of scalars. Note that j^* is conservative.

Proposition 7.1. *Let Q be a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3, and suppose the injective dimension of A as a k-module is finite. In this situation it holds for all $X \in {}_{Q,A}\mathrm{Mod}$ that*

(1) for all $G \in \mathrm{GPrj}_Q$ and $i > 0$, we have $\mathrm{Ext}^j_{Q,A}(G \otimes_k A, X) \cong \mathrm{Ext}^j_Q(G, j^*X)$.

(2) For all $H \in \mathrm{GInj}_Q$ and $i > 0$, we have that $\mathrm{Ext}^j_{Q,A}(X, \mathrm{Hom}_k(A, H)) \cong \mathrm{Ext}^j_Q(j^*X, H)$.

Proof. Let $G \in \mathrm{GPrj}_{Q,k}$ be a Gorenstein projective and $P_\bullet \to G$ be a projective resolution. We have that j^* is exact so j_* preserves projective object. That is $P_i \otimes_k A$ is projective for all $i \geqslant 0$. We claim $j_*P_\bullet$ is a resolution of j_*G. This can be checked pointwise in ${}_{Q,A}\mathrm{Mod}$. That is it suffices to see that

$$\ldots \to P_1(q) \otimes_k A \to P_0(q) \otimes_k A \to G(q) \otimes_k A \to 0$$

is exact in ${}_A\mathrm{Mod}$. Furthermore, since restriction of scalars is conservative and exact it suffices to check that the sequence is exact as a sequence of k-modules. It follows from Corollary 3.12 that $P_i(q)$ is projective for all q, since E_q is left adjoint to G_q and G_q is exact. In particular $P_\bullet(q) \to G(q)$ is a projective resolution. It follows that we have

$$H_i(P_\bullet(q) \otimes_k A) \cong \mathrm{Tor}^k_i(G(q), A).$$

Now from Corollary 4.5 we know that $G(q)$ is Gorenstein projective in ${}_k\mathrm{Mod}$, so it follows from [EJ11, Thm. 10.3.8 (9)] that

$$\mathrm{Tor}^k_i(G(q), A) \cong 0$$

since A has finite injective dimension as a k-module. It follows that $P_\bullet \otimes_k A \to G \otimes_k A$ is a projective resolution in ${}_{Q,A}\mathrm{Mod}$. The proof can now be completed by the following simple proof computation

$$\mathrm{Ext}^i_{Q,A}(G \otimes_k A, X) \cong H^i(\mathrm{Hom}_{Q,A}(P_\bullet \otimes_k A, X)) \cong H^i(\mathrm{Hom}_Q(P_\bullet, j^*X)) \cong \mathrm{Ext}^i_Q(G, j^*X).$$

Where the first isomorphism is by definition, the middle isomorphism is adjunction and the third is by definition.

The claim in (2) is proven dually. □

The next goal is to extend the results of Theorem 5.1 from the category of Q-shaped k-modules to the category Q-shaped category of A-modules. Firstly we will construct a 'lift' of the cotorsion pairs in Theorem 4.6, and then we will argue that these also satisfy the conditions of Hovey's theorem Theorem 2.22.

Definition 7.2. Let Q be a small ${}_k\mathrm{Mod}$-enriched category and A an k-algebra, then we define the *exact* objects, denoted $\mathcal{E}$, in ${}_{Q,A}\mathrm{Mod}$ to be the preimage of $\mathcal{L}_Q$ under $j^* \colon {}_{Q,A}\mathrm{Mod} \to {}_Q\mathrm{Mod}$. That is $\mathcal{E}$ is the full subcategory spanned by the objects $X \in {}_{Q,A}\mathrm{Mod}$ such that j^*X has finite projective, equivalently injective, dimension.

Theorem 7.3 (Thm. 4.4 [HJ21])**.** *Suppose Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3 and A has finite injective dimension as a k-module. In this situation*

(1) the pair $({}^\perp\mathcal{E}, \mathcal{E})$ is a cotorsion pair which is generated by the full subcategory, $\mathcal{M}$, of ${}_{Q,A}\mathrm{Mod}$ spanned by objects of the form $G \otimes_k A$, where $G \in \mathrm{GPrj}_Q$ is Gorenstein projective. Moreover, the cotorsion pair $({}^\perp\mathcal{E}, \mathcal{E})$ is hereditary and ${}^\perp\mathcal{E} \cap \mathcal{E} = \mathrm{Prj}_{Q,A}$.

(2) The pair $(\mathcal{E}, \mathcal{E}^\perp)$ is a cotorsion pair, which is cogenerated by the full subcategory, $\mathcal{M}'$, spanned by objects of the form $\mathrm{Hom}_k(A, H)$, where $H \in \mathrm{GInj}_k$ is Gorenstein injective in k. Moreover, the cotorsion pair $(\mathcal{E}, \mathcal{E}^\perp)$ is hereditary and $\mathcal{E} \cap \mathcal{E}^\perp = \mathrm{Inj}_{Q,A}$.

Furthermore, the subcategory $\mathcal{E}$ is thick.

Proof. We first prove that $\mathcal{E}$ is thick. Note that since j^* is exact and admit both adjoints it suffices to prove that $\mathcal{L}_Q$ is thick, which is proven in Proposition 4.7.

We now prove (1) as (2) can be proven analogously.

Note that to see that $({}^\perp\mathcal{E}, \mathcal{E})$ is a cotorsion pair, it suffices to prove that $({}^\perp\mathcal{E})^\perp \subseteq \mathcal{E}$. We can reduce to this, since by definition we have that $\mathcal{E} \subseteq ({}^\perp\mathcal{E})^\perp$. Furthermore, we have from Proposition 7.1 that $\mathcal{M} \subseteq {}^\perp\mathcal{E}$. This also proves that $\mathcal{M}^\perp = \mathcal{E}$.

Now suppose $X \in ({}^\perp\mathcal{E})^\perp$ we have to show $X \in \mathcal{E}$, which is equivalent to proving $j^*X \in \mathcal{L}_Q$. This is clear as we proved in Proposition 7.1 that

$$\mathrm{Ext}^i_Q(G, j^*X) \cong \mathrm{Ext}^i_{Q,\mathcal{A}}(G \otimes_k A, X)$$

for all $G \in \mathrm{GPrj}_Q$ Gorenstein projective. By assumption $G \otimes_k A \in \mathcal{M}$ so we have that

$$\mathrm{Ext}^i_{Q,A}(G \otimes_k A, X) \cong 0.$$

It follows that $j^*X \in \mathcal{L}_Q$, so we have that $X \in \mathcal{E}$. It also follows from this argument that $({}^\perp\mathcal{E}, \mathcal{E})$ is hereditary, since $(\mathrm{GPrj}_Q, \mathcal{L}_Q)$ is hereditary.

We now show that ${}^\perp\mathcal{E} \cap \mathcal{E} = \mathrm{Prj}_{Q,A}$. For "$\subseteq$", suppose $X \in {}^\perp\mathcal{E} \cap \mathcal{E}$ and choose a projective P with a surjection $P \to X$, taking the kernel we get a short exact sequence

$$0 \to Z \to P \to X \to 0.$$

We claim $P \in \mathcal{E}$. We may reduce to the case with $P \cong Q(q,-) \otimes_k A$. This follows since $(Q(q,-) \otimes_k A)_{q \in Q}$ are compact projective generators ${}_{Q,A}\mathrm{Mod}$ by Proposition 3.10 and the fact that $F_q(-) = Q(q,-) \otimes -$ is exact by Lemma 2.7 as $Q(q,-)$ is projective. It follows that there exists a set I and a surjection

$$\bigoplus_{i \in I} Q(q_i,-) \otimes_k A \to P \to 0$$

which is then a split epimorphism, since P is projective, the conclusion follows as $\mathcal{E}$ is closed under direct summands as j^* is exact and $\mathcal{L}_{Q,k}$ is closed under direct summands by classical theory.

For now we will denote

$$F_q \colon {}_A\mathrm{Mod} \to {}_{Q,A}\mathrm{Mod}$$

by F_q^A. Note that $Q(q,-) \otimes_k A = F_q^A(A)$, so we have to prove that $j^*(F_q^A(A)) \in \mathcal{L}_Q$. This can be seen as follows. Note that

$$j^*(F_q^A(A)) = F_q(j^*A)$$

and by assumption $j^*A = A$ has finite injective dimension as a k-module, so $F_q(j^*A) \in \mathcal{L}_Q$, since F_q is exact by Corollary 3.12 and admits a right adjoint, which is also exact, and thus preserves projective objects. It follows that $F_q^A(A) \in \mathcal{E}$. Now $\mathcal{E}$ is thick, so we see that $Z \in \mathcal{E}$. We also have that $X \in {}^\perp\mathcal{E}$ it follows that $\mathrm{Ext}^1_{Q,A}(X, Z) \cong 0$, so $X \oplus Z \cong P$, so it follows that X is projective.

For "$\supseteq$", note that $\mathrm{Prj}_{Q,A} \subseteq {}^\perp\mathcal{E}$ by definition, and $\mathrm{Prj}_{Q,A} \subseteq \mathcal{E}$ by the argument above. Therefore, we see that $\mathrm{Prj}_{Q,A} = {}^\perp\mathcal{E} \cap \mathcal{E}$.

Finally, to see that $({}^\perp\mathcal{E}, \mathcal{E})$ is hereditary we note that this is a special case of $\mathcal{E}$ being thick. To see this let $Y \in \mathcal{E}$ and we choose an injective with a monic $Y \xrightarrow{\phi} I$. Now $I \in \mathcal{E}$ since $({}^\perp\mathcal{E}, \mathcal{E})$ is a cotorsion pair. We now consider the short exact sequence

$$0 \to Y \xrightarrow{\phi} I \to \operatorname{coker}\phi \to 0.$$

It follows that $\operatorname{coker}\phi \in \mathcal{E}$, since $\mathcal{E}$ is thick. Finally, for any $X \in {}^\perp\mathcal{E}$ we may consider the long exact sequence

$$\mathrm{Ext}^1_{Q,A}(X, Y) \to \mathrm{Ext}^1_{Q,A}(X, I) \to \mathrm{Ext}^1_{Q,A}(X, \operatorname{coker}\phi) \to \mathrm{Ext}^2_{Q,A}(X, Y) \to \dots$$

we see that

$$\mathrm{Ext}^i_{Q,A}(X, \operatorname{coker}\phi) \cong \mathrm{Ext}^{i+1}_{Q,A}(X, Y)$$

since $I \in \mathrm{Inj}_{Q,A}$ is injective. Furthermore, since $\mathcal{E}$ is thick we have that $\mathrm{Ext}^1_{Q,A}(X, \operatorname{coker}\phi) \cong 0$, since $({}^\perp\mathcal{E}, \mathcal{E})$ is a cotorsion pair. Now this implies $\mathrm{Ext}^2_{Q,A}(X, Y) \cong 0$. Now choose an injective I' and a monomorphism $\psi \colon \operatorname{coker}\phi \to I'$, then $\operatorname{coker}\psi \in \mathcal{E}$, by the same argument as before. Hence $\mathrm{Ext}^1_{Q,A}(X, \operatorname{coker}\psi) \cong \mathrm{Ext}^2_{Q,A}(X, \operatorname{coker}\phi) \cong 0$. Which implies

$$\mathrm{Ext}^3_{Q,A}(X, Y) \cong \mathrm{Ext}^2_{Q,A}(X, \operatorname{coker}\phi) \cong 0.$$

Continuing inductively we see that

$$\mathrm{Ext}^i_{Q,A}(X, Y) \cong 0$$

for all $i > 0$. □

In order to apply Hovey's theorem again we need to show that the cotorsion pairs constructed above are complete. In order to show this we will take a slight detour into more general theory.

Definition 7.4. For $\mathcal{A}$ a Grothendieck category and $n \in \mathbb{N}$ we let $\mathcal{I}_n(\mathcal{A}) = \mathcal{I}_n$ denote the full subcategory spanned by object $X \in \mathcal{A}$ with $\mathrm{id}_{\mathcal{A}} X \leqslant n$.

Lemma 7.5. *Let $\mathcal{A}$ be a Grothendieck category generated by compact objects. For an object X and $n \geqslant 0$ one has $X \in \mathcal{I}_n$ if and only if for all finitely generated $F \in \mathcal{A}$ we have that*

$$\mathrm{Ext}^{n+1}_{\mathcal{A}}(F, X) \cong 0.$$

Proof. Let $X \to I^\bullet$ be an injective resolution of X and $\Omega^i(X) := (\ker(I^i \to I^{i+1}))$ be the ith cosyzygy. By definition, we have that $X \in \mathcal{I}_n$ if and only if $\Omega^n(X)$ is injective. By Baer's critirion [Kra98, Lem. 2.5] this is equivalent to

$$\mathrm{Ext}^1_{\mathcal{A}}(F, \Omega^n(X)) \cong 0$$

for all finitely generated $F \in \mathcal{A}$. Now by dimensions shifting we have that

$$\mathrm{Ext}^1_{\mathcal{A}}(F, \Omega^m(X)) \cong \mathrm{Ext}^{n+1}_{\mathcal{A}}(F, X)$$

proving the claim. □

Proposition 7.6. *Let $\mathcal{A}$ be a Grothendieck generated by compact objects and $n \in \mathbb{N}$.*

(1) If $\mathcal{A}$ has enough projectives, then $({}^\perp\mathcal{I}_n, \mathcal{I}_n)$ is a hereditary cotorsion pair, generated by a set. It follows that $({}^\perp\mathcal{I}_n, \mathcal{I}_n)$ is complete.
(2) If $\mathcal{A}$ is generated by a set of projective noetherian object, then $\mathcal{I}_n$ is closed under pure subobjects and pure quotients.

Proof. Ad (1), by assumption $\mathcal{A}$ is generated by a set $\mathcal{X}$ of compact objects. By definition an object is finitely generated if and only if it is a quotient of a finite direct sum of objects from $\mathcal{X}$. It follows that up to isomorphism there is only a set $\mathcal{F}$ of finitely generated objects. If we let $\Omega_n(\mathcal{F})$, be the full subcategory spanned by n'th syzygys of objects from $\mathcal{F}$. That is if $F \in \mathcal{F}$ and $P_\bullet \to F$ is a projective resolution of F then $\Omega_n(F) = \mathrm{coker}(P_{n+1} \to P_n)$ is in $\Omega(\mathcal{F})$. It follows from Lemma 7.5 and dimension shifting that

$$\Omega_n(\mathcal{F})^\perp = \mathcal{I}_n.$$

Now $\Omega_n(\mathcal{F})$ is small and hence the objects form a set. Thus it follows from [SS11, Lem. 2.15(3)] that $({}^\perp\mathcal{I}_n, \mathcal{I}_n)$ is complete. It is clear from the long exact sequence in $\mathrm{Ext}^*_{\mathcal{A}}(X, -)$ that $\mathcal{I}_n$ is coresolving, so it follows that $({}^\perp\mathcal{I}_n, \mathcal{I}_n)$ is hereditary as we have enough projectives.

Ad (2), Since $\mathcal{A}$ is generated by projective noetherian objects, so we know that any object is compact if and only if it is finitely generated if and only if it is noetherian. So for any $F \in \mathcal{F}$ and a projective resolution $P_\bullet \to F$ we may assume P_i is compact for all $i \geqslant 0$. It follows that $\Omega_n(F)$ is compact for all n. Suppose

$$0 \to X' \to X \to X'' \to 0$$

is a pure exact sequence, then for $F \in \mathcal{F}$ we get a long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\Omega_n(F), X') \to \mathrm{Hom}_{\mathcal{A}}(\Omega_n(F), X) \to \mathrm{Hom}_{\mathcal{A}}(\Omega_n(F), X'') \to \mathrm{Ext}^1_{\mathcal{A}}(\Omega_n(F), X') \to \dots$$

Now $\mathrm{Hom}_{\mathcal{A}}(\Omega_n(F), X) \to \mathrm{Hom}_{\mathcal{A}}(\Omega_n(F), X'')$ is surjective since $\Omega_n(F)$ is compact and the sequence is pure exact. It follows that

$$\mathrm{Ext}^1_{\mathcal{A}}(\Omega_n(F), X') \cong 0.$$

Again using dimension shifting it follows $X' \in \mathcal{I}_n$. Finally, $({}^\perp\mathcal{I}_n, \mathcal{I}_n)$ is resolving which implies X'' is in $\mathcal{I}_n$. □

Proposition 7.7. *If Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3 and A has finite injective dimension as a k-module. Then the cotorsion pair $({}^\perp\mathcal{E}, \mathcal{E})$ generated by a set and therefore complete.*

Proof. Note that for $\mathcal{A} = {}_Q\mathrm{Mod}$ Proposition 7.6 applies, so $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n)$ is a complete hereditary cotorsion pair, generated by a set, for all $n \geqslant 0$. Now for $n = \mathrm{FID}_Q$ the finite injective dimension of ${}_Q\mathrm{Mod}$ we have that $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n) = (\mathrm{GPrj}_Q, \mathcal{L}_Q)$. So there exists a set $\mathcal{G} \subseteq \mathrm{GPrj}_Q$ such that $\mathcal{G}^{\perp} = \mathcal{L}_Q$. It follows from Proposition 7.1 that $\{G \otimes_k A \mid G \in \mathcal{G}\}^{\perp} = \mathcal{E}$, so we have that $({}^{\perp}\mathcal{E}, \mathcal{E})$ is generated by a set and hence complete. □

This completes the proof that $({}^{\perp}\mathcal{E}, \mathcal{E})$ is complete. We can now replicate the proof from Theorem 5.1 to prove the existence of a projective model structure on ${}_{Q,A}\mathrm{Mod}$. Proving that $(\mathcal{E}, \mathcal{E}^{\perp})$ is complete requires somewhat more work. We will now work towards this goal.

Proposition 7.8. *If Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3, then for all $q \in Q$ the Q-shaped module $Q(q,-) \in {}_Q\mathrm{Mod}$ is noetherian. In particular, the category ${}_Q\mathrm{Mod}$ is generated by projective noetherian objects.*

Proof. We have to show that any subobject $I \to Q(q,-)$ is finitely generated. Note that by assumption $N_+(q) = \{p_1, \dots, p_n\}$. By assumption k is Gorenstein, so it is in particular noetherian. It follows, since $Q(q,p_i)$ is finitely generated, that the submodule $I(p_i)$ is finitely generated. Let $g_{i1}, \dots g_{i\ell(i)}$ denote the generators. We claim that the map

$$\sigma\colon \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{\ell(i)} Q(p_i,-) \to q(q,-)$$

induced by $Q(g_{ij},-)$, has image I. This can be checked pointwise so we are reduced to showing that

$$\sum_{i=1}^{n} \sum_{j=1}^{\ell(i)} \mathrm{Im}\, Q(g_{ij},p) \to I(p)$$

for every $p \in Q$.

For "$\subseteq$", this is exactly by construction, as for any $h \in Q(p_i,p)$ we have that $Q(g_{ij},p)(h) = hg_{ij}$ which is in $I(r)$ by definition.

For "$\supseteq$", we may reduce to $p = p_i$ for some i. If this was not the case $Q(q,p) = 0$ so $I(p) = 0$. Furthermore, we may reduce to show the claim on generators. This however is clear since

$$Q(g_{ij},p)(\mathrm{id}_{p_i}) = g_{ij}.$$

Proving the claim. □

Lemma 7.9. *The functor $j^*\colon {}_{Q,A}\mathrm{Mod} \to {}_Q\mathrm{Mod}$ preserves pure exact sequences.*

Proof. Let

$$\xi = 0 \to X' \to X \to X'' \to 0$$

be a pure short exact sequence. If $F \in {}_Q\mathrm{Mod}$ is compact, then $F \otimes_k A \in {}_{Q,A}\mathrm{Mod}$ is compact. This follows as

$$\mathrm{Hom}_{Q,A}(F \otimes_k A, -) \cong \mathrm{Hom}_Q(F, j^*(-))$$

and j^* preserves colimits as it is left adjoint to j_*. Now by assumption $\mathrm{Hom}_{Q,A}(F \otimes_k A, \xi)$ is exact, so it follows that $\mathrm{Hom}_Q(F, j^*(\xi))$ is exact. □

We will now input a result from [HJ21, Appendix A]. We do this is done in order to not discuss Kaplansky classes, which is a technical tool only used in showing completeness of $(\mathcal{E}, \mathcal{E}^{\perp})$, in details.

Theorem 7.10 (Thm. A.3 [HJ21])**.** *Let $\mathcal{M}$ be a Grothendieck k-linear category generated by compact objects. Let $\mathcal{F}$ be a full subcategory of $\mathcal{M}$. If*

(1) $\mathcal{F}$ is closed under pure quotients and pure subobjects.
(2) $\mathcal{F}$ is closed under coproducts in $\mathcal{M}$ and contains a generator of $\mathcal{M}$.

Then $(\mathcal{F}, \mathcal{F}^{\perp})$ is a complete cotorsion pair. In fact, every object in $\mathcal{M}$ has a $\mathcal{F}$-cover.

Theorem 7.11 ([HJ21, Thm.5.9]). *Let Q be a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3 and assume that the k-algebra has finite injective dimension as a k-module. In this situation, the cotorsion pair $(\mathcal{E}, \mathcal{E}^{\perp})$ is complete. In fact, the pair is* perfect. *That is every object in ${}_{Q,A}\mathrm{Mod}$ has a $\mathcal{E}$-cover and an $\mathcal{E}^{\perp}$-envelope.*

Proof. We prove that $\mathcal{F} = \mathcal{E}$ satisfies the assumptions of Theorem 7.11.

Ad (1), if $n = \mathrm{FID}$, then $\mathcal{L}_Q = \mathcal{I}_n$ so by Proposition 7.6 and Proposition 7.8 we know that $\mathcal{L}_Q$ is closed under pure quotients and pure subobjects. Therefore, it follows from Lemma 7.9 that $\mathcal{E}$ is closed under pure quotients and pure subobjects.

Ad (2), for $n = \mathrm{FID}$, then $\mathcal{L}_Q = \mathcal{I}_n$. It follows from [Ald+01, Cor. p. 163] that $\mathcal{I}_n$ is closed under filtered colimits. It follows that $\mathcal{E}$ is closed under filtered colimits, as $j^* \colon {}_{Q,A}\mathrm{Mod} \to {}_Q\mathrm{Mod}$ is left adjoint to j_* and thus preserves colimits. We proved in Theorem 7.3 that $\mathcal{E}$ is thick, so it follows that $\mathcal{E}$ is closed under coproducts.

It follows that $(\mathcal{E}, \mathcal{E}^{\perp})$ is complete and every object admits $\mathcal{E}$-cover by Theorem 7.10. It remains to show that every object admits an $\mathcal{E}^{\perp}$-envelope. The fact that $(\mathcal{E}, \mathcal{E}^{\perp})$ is complete contains the fact that every object admits a special $\mathcal{E}^{\perp}$-preenvelope, so it follows from the above argument and [Ald+01, Thm. 2.9] that every object has a $\mathcal{E}^{\perp}$-envelope. Which is what we wanted to prove. □

Theorem 7.12. *Let Q be any ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3 and suppose the k-algebra A has finite injective dimension as an k-module. In this situation we have that:*

1. *There exists an abelian model structure on ${}_{Q,A}\mathrm{Mod}$, where the cofibrant objects are given by the objects of ${}^{\perp}\mathcal{E}$, the trivial objects are given by the objects of $\mathcal{E}$ and every object is fibrant.*
2. *There exists an abelian model structure on ${}_{Q,A}\mathrm{Mod}$, where every object is fibrant, the trivial objects are given by the objects of $\mathcal{E}$ and the fibrant objects are given by the objects of $\mathcal{E}^{\perp}$.*

Proof. We prove (1) as (2) is proven analogously.

Similarly to the proof of Theorem 5.1 we claim that $({}^{\perp}\mathcal{E}, \mathcal{E}, {}_{Q,A}\mathrm{Mod})$ is a Hovey triple.

We proved in Theorem 7.3 that

$$({}^{\perp}\mathcal{E}, \mathcal{E} \cap {}_{Q,A}\mathrm{Mod}) = ({}^{\perp}\mathcal{E}, \mathcal{E})$$

is a cotorsion pair. Which is complete by Proposition 7.7.

Likewise, we proved in Theorem 7.3 that

$$({}^{\perp}\mathcal{E} \cap \mathcal{E}, {}_{Q,A}\mathrm{Mod}) = (\mathrm{Prj}_{Q,A}, {}_{Q,A}\mathrm{Mod}).$$

Which is a complete cotorsion pair, as ${}_{Q,A}\mathrm{Mod}$ has enough projectives by Proposition 3.4. □

Definition 7.13. If Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3 and the k-algebra A has finite injective dimension as a k-module. Then in accordance with Definition 5.3 we define the *projective* model structure on ${}_{Q,A}\mathrm{Mod}$ to be the model structure defined by (1) in Theorem 7.12 and define the *injective* model structure to be the model structure defined by (2) in Theorem 7.12.

Proposition 7.14. *Assume that Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3 and the k-algebra A has finite injective dimension as a k-module. In this situation, the model categories*

$$({}_{Q,A}\mathrm{Mod})_{\mathrm{Proj}} \qquad \text{and} \qquad ({}_{Q,A}\mathrm{Mod})_{\mathrm{Inj}}$$

have the same weak equivalences. In fact, for any map $\phi \colon X \to Y$ in ${}_{Q,A}\mathrm{Mod}$ the following are equivalent

1. *the map factors as $\phi = \pi\iota$, where ι is monic with $\operatorname{coker} \iota \in \mathcal{E}$ and π is epic with $\ker \pi \in \mathcal{E}$.*
2. *The map ϕ is a weak equivalence for the projective model structure on ${}_{Q,A}\mathrm{Mod}$.*
3. *The map ϕ is a weak equivalence for the injective model structure on ${}_{Q,A}\mathrm{Mod}$.*

Proof. The proof is exactly the same as the proof of Proposition 5.4. □

Definition 7.15. Let Q be a small ${}_k$Mod enriched category and suppose the k-algebra A has finite injective dimension as a k-module satisfying Setup 4.3, then the *Q-shaped derived category* of A is the category homotopy category

$$\mathcal{D}_Q(A) \coloneqq \mathrm{Ho}({}_{Q,A}\mathrm{Mod})$$

of ${}_{Q,A}$Mod with the projective model structure.

Theorem 7.16. *If Q is a small ${}_k$Mod-enriched category satisfying Setup 4.3, and A has finite injective dimension as a k-module. Then the category ${}^{\perp}\mathcal{E}$, respectively $\mathcal{E}^{\perp}$ is Frobenius with pro-injective objects given by $\mathrm{Prj}_{Q,A}$ and $\mathrm{Inj}_{Q,A}$, respectively. Furthermore, $\mathcal{D}_Q(A)$ admits a triangulated structure, such that the functors*

$${}^{\perp}\mathcal{E}/_{\mathrm{Prj}_{Q,A}} \to \mathcal{D}_Q(A) \qquad \text{and} \qquad \mathcal{E}^{\perp}/_{\mathrm{Inj}_{Q,A}} \to \mathcal{D}_Q(A)$$

are triangle equivalences.

Proof. The cotorsion pairs determining the projective, respectively injective model structure are all hereditary so the conclusion follows from Theorem 2.25 and Remark 2.26. ☐

We will now give a new criterion for determining weak equivalences of ${}_{Q,A}$Mod.

Proposition 7.17. *The adjunction*

$${}_{Q,A}\mathrm{Mod} \underset{j^*}{\overset{j_!}{\leftrightarrows}} {}_Q\mathrm{Mod}$$

is Quillen with respect to the projective model structure on ${}_{Q,A}$Mod and ${}_Q$Mod. Similarly, the adjunction

$${}_{Q,A}\mathrm{Mod} \underset{j_*}{\overset{j^*}{\rightleftarrows}} {}_Q\mathrm{Mod}$$

is Quillen with respect to the injective model structure on ${}_{Q,A}$Mod and ${}_Q$Mod. Furthermore, $j^*\colon {}_{Q,A}\mathrm{Mod} \to {}_Q\mathrm{Mod}$ is homotopical and if $\phi\colon X \to Y$ is a map in ${}_{Q,A}$Mod and $j^*\phi$ is a weak equivalence, then ϕ is a weak equivalence.

Proof. We prove the projective case, as the injective case is similar.

Suppose $\phi\colon X \to Y$ is a (trivial) fibration in ${}_{Q,A}$Mod, then as the projective model structure on ${}_{Q,A}$Mod is abelian we have that ϕ is an epimorphism with (trivially) fibrant kernel. Now j^* is exact so $j^*\phi\colon j^*X \to j^*Y$ is an epimorphism with $\ker j^*\phi \cong j^* \ker \phi$. Every object in ${}_Q$Mod is fibrant, so it remains to show that $\ker j^*\phi$ is trivial if ϕ is a trivial fibration. This is clear, since if ϕ is a trivial fibration, then $\ker \phi \in \mathcal{E}$, so $\ker j^*\phi \cong j^* \ker \phi \in \mathcal{L}_Q$. Proving the claim.

We now prove that j^* is homotopical. Note that by Proposition 7.14, the weak equivalences in the projective and injective model structures on ${}_{Q,A}$Mod coincide. Likewise, the weak equivalences in the projective and injective model structures on ${}_Q$Mod coincide by Proposition 5.4. Thus it follows by Ken Brown's lemma [Hov07, Lem. 1.1.12] that j^* preserves weak equivalences.

Finally, Suppose now that $\phi\colon X \to Y$ is a map in ${}_{Q,A}$Mod such that $j^*\phi$ is a weak equivalence in ${}_Q$Mod. We may factor $\phi = \pi\iota$, where ι is a cofibration and π is a trivial fibration in the projective model structure on ${}_Q$Mod. By the above $j^*\pi$ is a trivial fibration in the projective model structure on ${}_Q$Mod and $j^*\iota$ is a monomorphism since j^* is exact. Now by assumption $j^*\phi = j^*(\pi)j^*(\iota)$ is a weak equivalence and by the above argument so is $j^*\pi$. It follows by the 2-out-of-3 property for weak equivalence that $j^*\iota$ is a weak equivalence. This means by Proposition 5.4 that $\operatorname{coker} j^*\iota \in \mathcal{L}_Q$, so $\operatorname{coker} \iota \in \mathcal{E}$, so by Proposition 7.14 we have that ι is a weak equivalence. It follows from the 2-out-of-3 property for weak equivalences in ${}_{Q,A}$Mod that $\phi = \pi\iota$ is a weak equivalence. ☐

With this result in hand we are now able to introduce (co)homology and prove result analogous to the ones from Section 6.

Definition 7.18. Let $X \in {}_{Q,A}$Mod be a Q-shaped A-module, $q \in Q$ and $i > 0$. We define the i'th *(co)homology* of X at q to be the (co)homology of the underlying Q-shaped module. That is

$$H^i_{[q]}(X) \coloneqq H^i_{[q]}(j^*X) \qquad \text{and} \qquad H_i^{[q]}(X) \coloneqq H_i^{[q]}(j^*X).$$

Remark 7.19. Recall that in Definition 6.5 we defined the cohomology of X to be

$$H^i_{[q]}(X) = \mathbb{R}^i\mathrm{map}_Q(S\langle q\rangle, X).$$

This definition also makes sense in the setting of ${}_{Q,A}\mathrm{Mod}$ and is by definition an A-module. We claim that $j^*\mathrm{map}_Q(S\langle q\rangle, X) \cong \mathrm{map}_Q(S\langle q\rangle, j^*(X))$. To see this consider the computation

$$\begin{aligned} j^*\mathrm{map}_Q(S\langle p\rangle, X) &= j^*\left(\int_{q\in Q} X(q)^{S\langle p\rangle(q)}\right) \\ &\cong \int_{q\in Q} j^*\left(X(q)^{S\langle p\rangle(q)}\right) \\ &\cong \int_{q\in Q} j^*(X(q))^{S\langle p\rangle(q)} \\ &\cong \mathrm{map}_Q(S\langle p\rangle, j^*X). \end{aligned}$$

Here the first isomorphism follows because j^* is right adjoint to $j_!$ and therefore preserves ends. The second isomorphism follows from the fact that j^* preserves cotensors and the third is the definition. It follows, since j^* is exact, that the derived functors of $\mathrm{map}_Q(S\langle p\rangle, -)$ admit canonical A-module structures such that $j^*\mathbb{R}^i\mathrm{map}_Q(S\langle p\rangle, X) \cong \mathbb{R}^i\mathrm{map}_Q(S\langle p\rangle, j^*X)$. In particular, since j^* is conservative, both of the two possible definitions of cohomology in this setting agrees as to when maps are isomorphisms and when cohomology vanishes.

We will now extend Theorem 6.20 to the setting of ${}_{Q,A}\mathrm{Mod}$, this turns out to be quite simple given the results of this section.

Theorem 7.20. *Let Q be a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3* such that the pseudo-radical τ is nilpotent. Suppose the k-algebra A has finite injective dimension as a k-module and suppose k is hereditary. That is k is noetherian with* $\mathrm{gldim}\, A \leqslant 1$. *In this situation for all $X \in {}_{Q,A}\mathrm{Mod}$ the following are equivalent.*

(1) We have that X belongs to $\mathcal{E}$.
(2) For all $q \in Q$ and $i > 0$ we have that

$$H^i_{[q]}(X) \cong 0.$$

(3) For all $q \in Q$ we have that

$$H^1_{[q]}(X) \cong 0.$$

Similarly the following are also equivalent.

(1) We have X belongs to $\mathcal{E}$.
(2) For all $q \in Q$ and $i > 0$ we have that

$$H_i^{[q]}(X) \cong 0.$$

(3) For all $q \in Q$ we have that

$$H_1^{[q]}(X) \cong 0.$$

Proof. Note that since k is hereditary Theorem 6.20 applies to ${}_Q\mathrm{Mod}$ with the projective/injective model structure. In particular, we have that $H^i_{[q]}(X) = H^i_{[q]}(j^*X) \cong 0$ if and only if j^*X belongs to $\mathcal{L}_Q$, which is if and only if X is in $\mathcal{E}$. The claim for homology is proven analogously. □

Theorem 7.21. *Suppose Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3* such that the pseudo-radical τ is nilpotent. Suppose further, that the k-algebra A has finite injective dimension as a k-module and suppose k is hereditary. For a map $\phi\colon X \to Y$ in ${}_{Q,A}\mathrm{Mod}$ the following are equivalent.*

(1) The map ϕ is a weak equivalence.
(2) For all $q \in Q$ and $i > 0$ the map

$$H^i_{[q]}(\phi)\colon H^i_{[q]}(X) \to H^i_{[q]}(Y)$$

is an isomorphism.

(3) For all $q \in Q$ and $i \in \{1,2\}$ the map

$$H^i_{[q]}(\phi)\colon H^i_{[q]}(X) \to H^i_{[q]}(Y)$$

is an isomorphism.

Similarly, the following are equivalent

(1) The map ϕ is a weak equivalence.

(2) For all $q \in Q$ and $i > 0$ the map

$$H_i^{[q]}(\phi)\colon H_i^{[q]}(X) \to H_i^{[q]}(Y)$$

is an isomorphism.

(3) For all $q \in Q$ and $i \in \{1,2\}$ the map

$$H_i^{[q]}(\phi)\colon H_i^{[q]}(X) \to H_i^{[q]}(Y)$$

is an isomorphism.

Proof. Note that Theorem 6.21 applies as k is hereditary. We prove the claim for cohomology as the claim for homology can be proven analogously.

For (1) implies (2), if ϕ is a weak equivalence then $j^*\phi$ is a weak equivalence by Proposition 7.17. It follows by Theorem 6.21 that

$$H^i_{[q]}(\phi)\colon H^i_{[q]}(X) \to H^i_{[q]}(Y)$$

is an isomorphism for all $q \in Q$ and $i > 0$.

For (2) implies (3), this is trivial.

For (3) implies (1), If $\phi\colon X \to Y$ is a map in ${}_{Q,A}\mathrm{Mod}$ such that

$$H^i_{[q]}(\phi)\colon H^i_{[q]}(X) \to H^i_{[q]}(Y)$$

is an isomorphism for all $q \in Q$ and $i \in \{1,2\}$. It follows by Theorem 6.21 that $j^*\phi$ is a weak equivalence in ${}_Q\mathrm{Mod}$, and thus by Proposition 7.17 that ϕ is a weak equivalence. ☐

Proposition 7.22. *Suppose Q is a small ${}_k\mathrm{Mod}$-enriched category satisfying Setup 4.3* such that the pseudo-radical τ satisfies that $\tau^2 = 0$. Suppose A has finite injective dimension as a k-module and k is a principal ideal domain. In this situation for any map $\phi\colon X \to Y$ in ${}_{Q,A}\mathrm{Mod}$ the following are equivalent:*

(1) The map ϕ is a weak equivalence.

(2) The map

$$H^1_{[q]}(\phi)\colon H^1_{[q]}(X) \to H^1_{[q]}(Y)$$

is an isomorphism for every $q \in Q$.

(3) The map

$$H_1^{[q]}(\phi)\colon H_1^{[q]}(X) \to H_1^{[q]}(Y)$$

is an isomorphism for every $q \in Q$.

Proof. Since k is a principal ideal domain Theorem 6.26 applies, which proves the statement. ☐

8. Stable translation quivers and n-complexes

In this section we will present a few examples of small ${}_k\mathrm{Mod}$-enriched categories satisfying Setup 4.3*. Corresponding to the results of [HJ21, Section 8] and the example of n-complexes. This will be mostly without proofs, but with ample references. We do it this way because the focus of this thesis was on extending the theory developed in [HJ21] to the class of all locally Gorenstein categories.

Definition 8.1. A *quiver* is an ordered quadruple $(\Gamma_0, \Gamma_1, s\colon \Gamma_1 \to \Gamma_0, t\colon \Gamma_1 \to \Gamma_0)$. With $\Gamma_0, \Gamma_1 \in \mathrm{Set}$. We say that Γ_0 is the set of *vertices* and Γ_1 is the set of *edges*. The maps s, t are called the *source* and *target* maps, respectively.

We say that a $\Gamma = (\Gamma_0, \Gamma_1, s, t)$ is *locally finite* if for every $q \in \Gamma_0$ the set $t^{-1}(\{q\})$ is finite.

Remark 8.2. Combinatorially minded readers might argue that quivers are nothing, but directed graphs. We however prefer this name as we equivalently could consider a quiver a functor

$$\Gamma \colon \mathcal{Q} \to \mathrm{Set}$$

where $\mathcal{Q}$ is the category with to objects V, E and four morphisms $\mathrm{Id}_V, \mathrm{Id}_E, s \colon E \to V$ and $t \colon E \to V$. We define the category of quivers Quiv to be the functor category:

$$\mathrm{Quiv} \coloneqq \mathrm{Fun}(\mathcal{Q}, \mathrm{Set}).$$

Definition 8.3. Let $\Gamma = (\Gamma_0, \Gamma_1, s, t)$ be a quiver, then a *path* in Γ is a collection $(e_i)_{i=1}^n$ of edges $e_i \in \Gamma_1$, such that $t(e_i) = s(e_{i+1})$ for every $i \geqslant 1$.

Define the *path category* on Γ to be the category, $P\Gamma$, with objects Γ_0 and for $p, q \in \Gamma_0$ the set of maps from from p to q is given by

$$P\Gamma(p,q) \coloneqq \{(e_i)_{i=1}^n \mid n \in \mathbb{N} \text{ and } (e_i)_{i=1}^n \text{ is a path in } \Gamma \text{ from } p \text{ to } q\}.$$

and if $p = q$ we set

$$P\Gamma(p,p) \coloneqq \{(e_i)_{i=1}^n \mid n \in \mathbb{N} \text{ and } (e_i)_{i=1}^n \text{ is a path in } \Gamma\} \cup \{\mathrm{id}_p\}.$$

Here composition is given by concatenation of paths. We will typically write $\Gamma(p,q) \coloneqq P\Gamma(p,q)$.

Proposition 8.4 (Page 1-6 [GZ12])**.** *The action of sending a quiver Γ to its path category $P\Gamma$ extends to a functor $P \colon \mathrm{Quiv} \to \mathrm{Cat}$ and the functor is left adjoint to the functor*

$$U \colon \mathrm{Cat} \to \mathrm{Quiv}$$

sending a small category to its underlying quiver.

Remark 8.5. Recall that for any commutative ring R the free R-module functor

$$\mathrm{Set} \to {}_R\mathrm{Mod}$$

admits a symmetric monoidal structure with respect to the cartesian product on Set and tensor product of R-modules. It follows that there is a functor

$$R[-] \colon \mathrm{Cat} \to {}_R\mathrm{Mod}\text{-}\mathrm{Cat}$$

given by sending a category $\mathcal{C}$ to the ${}_R\mathrm{Mod}$-enriched category $R\mathcal{C}$ with objects $\mathrm{Ob}R\mathcal{C} = \mathrm{Ob}\mathcal{C}$ and for objects $p, q \in R\mathcal{C}$ the R-module of maps from p to q is the free R-module on the set $\mathrm{Hom}_{\mathcal{C}}(p,q)$. In fact, the functor $R[-]$ is left adjoint to the underlying category functor

$$U \colon {}_R\mathrm{Mod}\text{-}\mathrm{Cat} \to \mathrm{Cat}.$$

Definition 8.6. Let Γ be a quiver and R be a commutative ring, then we define the *path category over R* to be the ${}_R\mathrm{Mod}$-enriched category

$$R\Gamma \coloneqq R[P\Gamma].$$

This already provides us with a way to consider R-linear representations of a quiver Γ. Namely, we consider R-linear functors $R\Gamma \to \mathcal{A}$ for some ${}_R\mathrm{Mod}$-enriched category $\mathcal{A}$. Now in general $R\Gamma$ will not satisfy all of Setup 4.3* and therefor we will consider additional structure on our quiver, which in some cases allow us to consider smaller categories which does satisfy Setup 4.3*

Definition 8.7. A *stable translation quiver* is a triple $(\Gamma, \rho \colon \Gamma_0 \to \Gamma_0, \sigma \colon \Gamma_1 \to \Gamma_1)$ where Γ is a quiver, and ρ and σ are bijections such that for every $\alpha \colon p \to q$ an edge in Γ, the arrow $\sigma(\alpha) \colon \rho(q) \to p$ has source $\rho(q)$ and target p. We normally call ρ a *translation* and σ a *semi-translation.*

Example 8.8. We will give a few ways to construct examples of stable translation quivers on a quiver Γ.

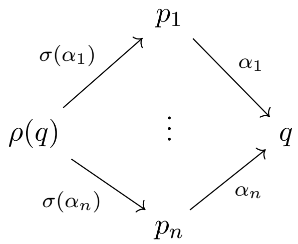
(1) The *double quiver*, Γ^{dou}, of Γ is stable translation quiver. The double quiver is given by $\Gamma_0^{\mathrm{dou}} = \Gamma_0$ and $\Gamma_1^{\mathrm{dou}} = \Gamma_1 \coprod \Gamma_1^{\mathrm{op}}$. Here $\Gamma_1^{\mathrm{op}} \coloneqq \{x^* \colon p \to q \mid x \colon q \to p \in \Gamma_1\}$. The translation is given by the identity on Γ_0 and the semi-translation is given by $\sigma(x) = x^*$ for $x \in \Gamma_1$ and $\sigma(x^*) = x$.

(2) The *repetitive quiver*, Γ^{rep}, is a stable translation quiver. It is given by $\Gamma_0^{\text{rep}} = \Gamma_0 \times \mathbb{Z}$ and has edges

$$x_i : (p,i) \to (q,i) \qquad \text{and} \qquad x_i^* : (q,i) \to (p,i-1)$$

for every $x : p \to q$ and edge in Γ. The translation is given by $\rho(p,i) = (p,i+1)$ for every $(p,i) \in \Gamma \times \mathbb{Z}$ and $\sigma(x_i) = x_{i+1}^*$ and $\sigma(x_i^*) = x_i$.

Definition 8.9. If (Γ,ρ,σ) is a locally finite stable translation quiver and $q \in \Gamma$ is a vertex. Then we define the *mesh* at q to be the diagram



where $\{p_1, \ldots p_n\} = t^{-1}(\{q\})$.

Definition 8.10. Let (Γ,ρ,σ) be a locally finite translation quiver and R a commutative ring. The *mesh ideal*, $\mathfrak{m}$, in $R\Gamma$ is the two-sided ideal in $R\Gamma$ generated the mesh at $q \in \Gamma$ for every q. That is, it is generated by $\langle \sigma(\alpha_1), \ldots, \sigma(\alpha_n) \rangle \subseteq R\Gamma(\rho(q), q)$ for every $q \in \Gamma_0$.

We let define the *mesh category*, $R\langle\Gamma\rangle$, of a stable translation quiver (Γ,ρ,σ) (over R) to be the ${}_R\text{Mod}$-enriched category

$$R\langle\Gamma\rangle := R\Gamma/\mathfrak{m}.$$

Proposition 8.11 ([HJ21, Lem.8.6])**.** *If (Γ,ρ,σ) is a locally finite stable translation quiver, then the R-linear mesh category $R\langle\Gamma\rangle$ satisfies condition* (4)$*$ *in Setup 4.3*. With the ideal given by the arrow ideal τ. That is $\tau(p,q)$ is the submodule of $R\langle\Gamma\rangle(p,q)$ generated by the edges of Γ.*

The R-linear path category is typically way to big to satisfy Setup 4.3*. However the R-linear mesh category might not be. In the next few results we will see that in some cases the mesh category will satisfy Setup 4.3*.

Definition 8.12. Let $n \in \mathbb{N}$ be an integer. Then the we let $\mathbb{A}_n$ by the quiver with

$$(\mathbb{A}_n)_0 = [n] = \{1, \ldots, n\}$$

and a single edge $i \to i+1$ for every $i \in [n-1]$.

Theorem 8.13 ([HJ21, Thm. 8.8])**.** *If R is any commutative ring, then for any $n \in \mathbb{N}$ the* ${}_R$Mod*-enriched category $R\langle(\mathbb{A}_n)^{\text{dou}}\rangle$ satisfies Setup 4.3*. More precisely, the following hold:*

(1) For every $p,q \in (\mathbb{A}_n)^{\text{dou}}$, the R-module $R\langle(\mathbb{A}_n)^{\text{dou}}\rangle(p,q)$ is free of dimension $\min(p,q,n+1-p,n+1-q)$.

(2) The functor $\mathbb{S} : R\langle(\mathbb{A}_n)^{\text{dou}}\rangle \to R\langle(\mathbb{A}_n)^{\text{dou}}\rangle$ given by $\mathbb{S}(q) = n+1-q$ on objects and $\mathbb{S}(a_q) = (-1)^q a_{n-q}^*$ and $\mathbb{S}(a_q) = (-1)^{n-q} a_{n-q}$ on arrows is a Serre functor.

Moreover, the arrow ideal τ is nilpotent with $\tau^n = 0$.

The above theorem implies that the category of representations of the double quiver of $\mathbb{A}_n$ may be studied with the methods of this thesis. The next implies that the same holds for the repetitive quiver on $\mathbb{A}_n$.

Theorem 8.14 ([HJ21, Thm. 8.11])**.** *If R is any commutative ring, then for any $n \in \mathbb{N}$ the* ${}_R$Mod*-enriched category $R\langle(\mathbb{A}_n)^{\text{rep}}\rangle$ satisfies Setup 4.3*. More precisely, the following hold:*

(1) The R-module $R\langle(\mathbb{A}_n)^{\text{rep}}\rangle((p,i),(q,j))$ is free of rank 0 *or* 1.

(2) The functor $\mathbb{S} : R\langle(\mathbb{A}_n)^{\text{rep}}\rangle \to R\langle(\mathbb{A}_n)^{\text{rep}}\rangle$ given by $\mathbb{S}(p,i) = (n+1-p, i+1-p)$ on objects and $\mathbb{S}(a_{p,i}) = (-1)^p a_{n-p,i+1-p}$ and $\mathbb{S}(a_{p,i}^*) = (-1)^{n-p} a_{n-p,i-p}$ is a Serre functor.

Furthermore, the arrow ideal τ is nilpotent with $\tau^n = 0$.

Finally, we will discuss the case of n-complexes.

Definition 8.15. For $n \geqslant 2$ consider the ${}_R\mathrm{Mod}$-enriched category, $\mathcal{O}_n$, with objects $\mathbb{Z}$ and mapping R-module between p and q in $\mathbb{Z}$ given by

$$\mathrm{Hom}_{\mathcal{O}_n}(p,q) = \begin{cases} R & \text{if } p \geqslant q \text{ and } p-q<n \\ 0 & \text{else.} \end{cases}$$

If $\mathcal{A}$ is an R-linear category, then we define the category of *n-complexes* in $\mathcal{A}$ as

$$\mathrm{Ch}_n(\mathcal{A}) := \mathrm{Fun}^R(\mathcal{O}_n, \mathcal{A}).$$

Remark 8.16. Note that when $n = 2$ the category of n-complexes is the category of chain complexes in $\mathcal{A}$.

Theorem 8.17. *Let R be a commutative ring and $n \geqslant 2$. In this situation the ${}_R\mathrm{Mod}$-enriched category $\mathcal{O}_n$ satisfies Setup 4.3*. More precisely, the following hold:*

1. *For every $p, q \in \mathcal{O}_n$ the R-module $R\langle \mathcal{O}_n \rangle$ is free of rank 0 or 1 and it is free of rank 1 exactly when $p \geqslant q$ and $p-q<n$.*
2. *The functor $\mathbb{S}\colon \mathcal{O}_n \to \mathcal{O}_n$ given by $\mathbb{S}(p) = p+n-1$ and for $\alpha\colon p \to q$ it is given by*
$$\mathbb{S}(\alpha) = \alpha$$
is a Serre functor.

More over the zero ideal serves as the pseudo-radical, which is of course nilpotent.

Proof. The only thing which is not immediately clear is the fact that $\mathbb{S}$ is a Serre functor. To see this consider $p, q \in \mathcal{O}_n$. We have to show that

$$\mathrm{Hom}_{\mathcal{O}_n}(p,q) \cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathcal{O}_n}(\mathbb{S}(q),p), R).$$

First of all, we note that if $p \geqslant q$ and $p-q<n$ then $\mathbb{S}(q) = q+n-1 > p-1$ so $\mathbb{S}(q) \geqslant p$. Furthermore, we have that

$$\mathbb{S}(q) - p < n.$$

It follows that $\mathrm{Hom}_{\mathcal{O}_n}(\mathbb{S}(q),p) \cong R$. In particular, if $p \geqslant q$ and $p-q<n$

$$\mathrm{Hom}_{\mathcal{O}_n}(p,q) \cong R \cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathcal{O}_n}(\mathbb{S}(q),p)).$$

The last part is to show that this is natural in p, q. This is however a standard exercise in diagram chasing. □

The above result implies that the results of the rest of this thesis may be applied to study n-complexes. Finally, we will compare cohomology as defined in Section 6, to so called mesh homology.

Definition 8.18. Suppose (Γ, ρ, σ) locally finite stable translation quiver, k is a Gorenstein commutative ring and A is a k-algebra with finite injective dimension and $X \in {}_{k\langle\Gamma\rangle,A}\mathrm{Mod}$ is a $k\langle\Gamma\rangle$-shaped complex in ${}_A\mathrm{Mod}$. In this situation we define the *mesh homology*, $\mathcal{H}_q(X)$, at $q \in \Gamma$ is the homology of the complex

$$X(\rho(q)) \to \bigoplus_{i=1}^{n} X(p_i) \to X(q)$$

induced by the mesh at q.

Furthermore, we say that (Γ, ρ, σ) is *normal* if

$$\mathcal{H}_q(k\langle\Gamma\rangle(p,-)) \cong 0$$

for every $p, q \in \Gamma$.

Proposition 8.19 ([HJ21, Prop. 8.18])**.** *If (Γ, ρ, σ) is a locally finite and normal stable translation quiver, then we have that for every $X \in {}_{k\langle\Gamma\rangle,A}\mathrm{Mod}$ we have that*

$$\mathcal{H}_q(X) \cong H_1^{[q]}(X)$$

naturally in X for every $q \in \Gamma$.

Proof sketch. We want to leverage that $\operatorname{Tor}_i^Q(-,-)$ is balanced and use that

$$k\langle\Gamma\rangle(-,\rho(q)) \to \bigoplus_{i=1}^{n} k\langle\Gamma\rangle(-,p_i) \to k\langle\Gamma\rangle(-,q) \to S\{q\} \to 0$$

is the beginning of a projective resolution of $S\{q\} \in \operatorname{Mod}_Q$. Given this the statement thus follows from the coYoneda lemma, which gives that

$$k\langle\Gamma\rangle(-,m) \otimes_{k\langle\Gamma\rangle} X \cong X(m)$$

for every $m \in \Gamma$. Which in turns identifies the first homology of X at q with the mesh homology at q.

Finally to see that this is an exact sequence, note that normality gives exactness on the left, and the fact that $t^{-1}(q) = \{p_1, \ldots, p_n\}$ gives that the image of

$$\bigoplus_{i=1}^{n} k\langle\Gamma\rangle(-,p_i) \to k\langle\Gamma\rangle(-,q)$$

is exactly the arrow ideal. Which has been shown to be the pseudo-radical in Proposition 8.11. Finally, note that this is a projective resolution as representable functors are projective, by the Yoneda lemma. □

References

[Kel82] Max Kelly. *Basic concepts of enriched category theory*. Vol. 64. CUP Archive, 1982.

[Wei95] Charles A Weibel. *An introduction to homological algebra*. 38. Cambridge university press, 1995.

[Kra98] Henning Krause. "Exactly definable categories". In: *Journal of Algebra* 201.2 (1998), pp. 456–492.

[Ald+01] S Tempest Aldrich et al. "Covers and envelopes in Grothendieck categories: flat covers of complexes with applications". In: *Journal of Algebra* 243.2 (2001), pp. 615–630.

[Hov02] Mark Hovey. "Cotorsion pairs, model category structures, and representation theory". In: *Mathematische Zeitschrift* 241.3 (2002), pp. 553–592.

[KS05] M. Kashiwara and P. Schapira. *Categories and Sheaves*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2005. ISBN: 9783540279495. URL: https://books.google.dk/books?id=K-SjOw%5C_2gXwC.

[Hov07] Mark Hovey. *Model categories*. 63. American Mathematical Soc., 2007.

[EEG08] Edgar Enochs, Sergio Estrada, and JR Garcia–Rozas. "Gorenstein categories and Tate cohomology on projective schemes". In: *Mathematische Nachrichten* 281.4 (2008), pp. 525–540.

[Hir09] Philip S Hirschhorn. *Model categories and their localizations*. 99. American Mathematical Soc., 2009.

[Lur09] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009.

[EJ11] Edgar E Enochs and Overtoun MG Jenda. "Relative homological algebra". In: *Relative Homological Algebra*. de Gruyter, 2011.

[SS11] Manuel Saorin and Jan Stovicek. "On exact categories and applications to triangulated adjoints and model structures". In: *Advances in Mathematics* 228.2 (2011), pp. 968–1007.

[GZ12] Peter Gabriel and Michel Zisman. *Calculus of fractions and homotopy theory*. Vol. 35. Springer Science & Business Media, 2012.

[Mac13] Saunders Mac Lane. *Categories for the working mathematician*. Vol. 5. Springer Science & Business Media, 2013.

[Lor15] Fosco Loregian. "This is the (co) end, my only (co) friend". In: *arXiv preprint arXiv:1501.02503* (2015).

[Gil16] James Gillespie. "Hereditary abelian model categories". In: *Bulletin of the London Mathematical Society* 48.6 (2016), pp. 895–922.

[DSS17] Ivo Dell'Ambrogio, Greg Stevenson, and Jan Stoviecek. "Gorenstein homological algebra and universal coefficient theorems". In: *Mathematische Zeitschrift* 287.3 (2017), pp. 1109–1155.

[HJ21] Henrik Holm and Peter Jorgensen. "The Q-shaped derived category of a ring". In: *arXiv preprint arXiv:2101.06176* (2021).

[Lan21] Markus Land. *Introduction to Infinity-Categories*. Springer Nature, 2021.