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Tannaka Duality

On Tannaka duality for symmetric fusion categories

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Abstract

In this project out of course scope we present the theory needed to state the Tannaka duality for symmetric fusion categories. Specifically that if $\mathscr{A}$ is a symmetric fusion category and $\mathscr{A}$ admits a fiber functor $\Phi\colon \mathscr{A} \to \mathrm{Vect}_{\mathbb{C}}$ then we have an monoidal equivalence $\mathscr{A} \simeq \mathrm{Rep}(\mathrm{Aut}^{\otimes} \Phi)$. In particular we prove that for a finite group G, the category of finite dimensional representations of G, $\mathrm{Rep}\, G$, is a symmetric fusion category. This is mainly done by lifting properties of $\mathrm{Vect}_{\mathbb{C}}$ to $\mathrm{Rep}\, G$.

The project starts by discussing the tensor product of vector spaces both through construction and through its universal property. We then discuss k-algebras and show that these are naturally thought of as the monoid objects in Vect_k. Finally we show an equivalence of the categories $\mathrm{Rep}\, G \simeq \mathbb{C}[G]\text{-Mod}(\mathrm{Vect}_{\mathbb{C}})$. We then abstract from the theory of the tensor product of vector spaces, to the theory of monoidal categories. In particular we introduce monoidal functors and monoidal natural transformations, braided and symmetric monoidal categories and braided monoidal functors. We then introduce dualizable objects and rigid monoidal categories. We then turn to enriched categories, in particular we discuss Abelian categories, simple objects in abelian categories and semi simple abelian categories. Here a noteworthy result we prove is Schur's Lemma. Which specialize to the case of linear categories in a particularly nice way. Namely that For a simple object X in a linear category $\mathrm{End}(X) \cong \mathbb{C}$. Then to state the theorem of Tannaka duality for symmetric fusion categories we then combine the notions of monoidal categories and linear categories, by requiring compatibility between the two. This defines tensor categories and fusion categories.

The final part of this project will be dedicated to proving a slightly weaker statement. To prove this statement we prove The Tannaka reconstruction theorem of $\mathbb{C}$-algebras and use this to prove the Tannaka duality for representations of a finite group G.

Introduction and motivation

The goal of this project out of course scope is to showcase the work i have done with my advisor Thomas in the third and fourth term of the second year of my bachelors degree.

In this project we present the prerequisites to the theory of fusion categories with the goal of presenting the statement of Tannaka duality for symmetric fusion categories first proved by Deligne. This theorem answers a very rudimentary type of questions in mathematics to which i will give an analogy. It is easy to see that the finite direct sum cyclic groups is an finitely generated abelian group. One might then to think ask the question are all finitely generated abelian groups isomorphic to some finite direct sum of cyclic groups. This is of course a well known fact. In a similar fashion one can show the category of representations of a finite group G is a symmetric fusion category. The question to be answered is then do all symmetric fusion categories arise this way and the answer given by Deligne in 1990 is yes. However before we are able ask this question in a rigid manor, we will need both vocabulary and theory. We will present the theory to stringently ask this question.

We will assume that the reader is familiar with elementary notions from category, linear algebra and group theory. In particular we will assume that the reader are comfortable with universal properties, various (co)limits and basic examples of these. Standard textbooks and references for the subjects presented in this project would be [Mac13], [nLa], [Wei95], [Kel82], [Eti+16], [Rie14] and [Tel05]. The project is mostly self contained with only theorems of little significants to this project proofed by reference.

One unfortunate thing lacking from this project is a proper chapter on string diagram formalism. I have written an appendix, this is however quite incomplete. For a more complete introduction i propose the paper [Bar15].

Notation. We use a couple of conventions in this project in particular $\simeq$ will always mean an equivalence of categories. $A \cong B$ will mean that A and B are isomorphic in some category. When handling natural transformation we will only denote components of the natural transformation with indices if we fear the lack thereof might further complicate the proof.

Contents

1 Tensor products and algebras in Vect_k

Given two vector spaces V, W over a field k, one generally has multiple ways of constructing new vector spaces from these. One reoccurring construction will be the direct sum $V \oplus W$, of V and W. Another one is the tensor product $V \otimes W$ - this section will concern itself with tensor products, both through the direct construction and through the universal property of tensor products. Following this we will define algebras in the category of vector spaces over a field k and compare the categorical and classical notion of these.

1.1 Construction of the tensor product

We now concretely construct the tensor products of two vector spaces over a field k.

Definition 1.1. Let $V, W \in \mathrm{Vect}_k$, we define the tensor product $V \otimes W$ to be as following

$$V \otimes W := F(V \times W)/\sim,$$

Where $F(A)$ denotes the free vector space with basis A and is the equivalence relation "$\sim$" is generated by the relation that for all $a, b \in V$ and $c, d \in W$ and $r \in k$.

$$\begin{aligned}(a+b, c) &\sim (a, c) + (b, c),\\ (a, c+d) &\sim (a, c) + (a, d),\\ r(a, c) &\sim (ra, c), \text{ and}\\ r(a, c) &\sim (a, rc).\end{aligned}$$

We denote the equivalence class of $(v, w) \in V \times W$ by $v \otimes w$. Additionally if $V, W, S, T \in \mathrm{Vect}_k$ and $f : V \to W$, $g : S \to T$ are linear maps then one can define the tensor product map by

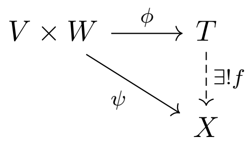
$$\begin{aligned}f \otimes g : V \otimes S &\to W \otimes T\\ v \otimes s &\longmapsto f(v) \otimes g(s).\end{aligned}$$

This construction yields a linear map and will be used multiple times in the following chapters, and is essential to multiple proofs in this chapter.

1.2 The universal property of the tensor products

Now we introduce the universal property of tensor products and show that the tensor product of vector spaces satisfy a plethora of properties.

Definition 1.2. If $V, W, T \in \mathrm{Vect}_k$ and $\phi : V \times W \to T$ is a bilinear map we say that (T, ϕ) is the *tensor product* of V and W if for every vector space $X \in \mathrm{Vect}_k$ and bilinear map $\psi : V \times W \to X$ there exist a unique linear map $f : T \to X$ such that the following diagram



commutes.

Using this definition its easy to see that T is unique up to isomorphism, and we will state without proof that the tensor product defined earlier satisfies this universal property. It is easy to see $V \otimes W \cong W \otimes V$ by the linear map $\tau_{V,W}$ where $v \otimes w \mapsto w \otimes v$. We are now able to prove a few propositions about the tensor product.

Remark. Given a pair of vector spaces $(V, W) \in \mathrm{Vect}_k \times \mathrm{Vect}_k$ the procedure of assigning their tensor product, and taking pairs of linear maps and assigning their tensor product of maps is a bifunctor

$$\otimes : \mathrm{Vect}_k \times \mathrm{Vect}_k \to \mathrm{Vect}_k\,.$$

Proposition 1.3. *For all $U, V, W \in \mathrm{Vect}_k$ their exists an isomorphism, called the associator,*

$$\alpha_{UVW} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

natural in all three arguments.

Proof. Let $U, V, W \in \mathrm{Vect}_k$, and define the following maps $\phi : U \times V \times W \to (U \otimes V) \otimes W$ given by $\phi(x, y, z) = (x \otimes y) \otimes z$ and $\psi : (U \times V \times W \to U \otimes (V \otimes W)$ by $\psi(x, y, z) = x \otimes (y \otimes z)$. Then by the universal property of the tensor product their exist a linear map $\alpha_{UVW} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ such that

$$\begin{array}{ccc} (U \times V) \times W & \xrightarrow{\phi} & (U \otimes V) \otimes W \\ & \searrow^{\psi} & \downarrow{\scriptstyle \alpha_{UVW}} \\ & & U \otimes (V \otimes W) \end{array}$$

commutes. One can obtain a linear map $\alpha^{-1}_{UVW} : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ in a similar fashion. It is routine to check that these are mutually inverse. We now check that α_{UVW} is natural in U, V and W. Let $U, V, W, U', V', W' \in \mathrm{Vect}_k$ and $f : U \to U'$, $g : V \to V'$ and $h : W \to W'$ be linear maps. Thus for $(a \otimes b) \otimes c \in (U \otimes V) \otimes W$ we see that

$$\begin{aligned} (\alpha_{U'V'W'}((f \otimes g) \otimes h))((a \otimes b) \otimes c) =& \alpha_{U'V'W'}((f(a) \otimes g(b)) \otimes h(c)) \\ = f(a) \otimes (g(b) \otimes h(c)) =& (f \otimes (g \otimes h))\alpha_{UVW}((a \otimes b) \otimes c) \end{aligned}$$

Showing that α is indeed a natural isomorphism. □

The tensor product of vector spaces also admit two more natural isomorphisms of special interest.

Proposition 1.4. *For all $V \in \mathrm{Vect}_k$ their exists isomorphisms $l : k \otimes V \to V$ and $r : V \otimes k \to V$, called the left and right unitors, natural in V.*

Proof. Let $V \in \mathrm{Vect}_k$ then define the map $l : k \otimes V \to V$ by the function $k \otimes V \ni r \otimes x \mapsto rx \in V$. This is clearly a linear map, it is the extension of the scalar multiplication of V by the tensor product and given the data $V \xrightarrow{f} W$ in Vect_k it holds that

$$l_W(\mathrm{id}_k \otimes f(r \otimes x)) = l_W(r \otimes f(x)) = rf(x) = f(rx) = f(l_V(r \otimes x)).$$

Showing that l is natural in V. The construction and subsequent proof for the right unitor is analogous and therefor omitted. □

This shows that k in some sense is the unit with respect to tensoring on the left and right.

Theorem 1.5 (Coherence theorem)**.** *For all $A, B, C, D \in \mathrm{Vect}_k$ the following diagrams commute*

$$\begin{array}{ccccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{ABC} \otimes \mathrm{id}_D} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{AB \otimes CD}} & A \otimes ((B \otimes C) \otimes D) \\ \downarrow{\scriptstyle \alpha_{A \otimes BCD}} & & & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \alpha_{BCD}} \\ (A \otimes B) \otimes (C \otimes D) & & \xrightarrow{\qquad \alpha_{ABC \otimes D} \qquad} & & A \otimes (B \otimes (C \otimes D)) \end{array}$$

and

$$\begin{array}{ccc} (A\otimes k)\otimes B & \xrightarrow{\alpha_{AkB}} & A\otimes(k\otimes B) \\ & {\scriptstyle r\otimes \mathrm{id}_B}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_A\otimes l} & \\ & A\otimes B & \end{array}$$

Remark. These diagrams are usually known as the pentagon and triangle diagrams.

Proof. Let $A, B, C, D \in \mathrm{Vect}_k$ then for $((a\otimes b)\otimes c)\otimes d \in ((A\otimes B)\otimes C)\otimes D$ we see by applying the associators that

$$\begin{aligned} &(\alpha_{ABC\otimes D}\alpha_{A\otimes BCD})(((a\otimes b)\otimes c)\otimes d) = \alpha_{ABC\otimes D}((a\otimes b)\otimes(c\otimes d)) \\ &= a\otimes(b\otimes(c\otimes d)) = \mathrm{id}_A\otimes\alpha_{BCD}(a\otimes((b\otimes c)\otimes d)) \\ &= (\mathrm{id}_A\otimes\alpha_{BCD}\circ\alpha_{AB\otimes CD})((a\otimes(b\otimes c))\otimes d) = (\mathrm{id}_A\otimes\alpha_{BCD}\circ\alpha_{AB\otimes CD}\circ\alpha_{ABC}\otimes\mathrm{id}_D)(((a\otimes b)\otimes c)\otimes d) \end{aligned}$$

showing the commutativity of the pentagon diagram. Similarly for $(a\otimes r)\otimes b \in (A\otimes k)\otimes B$ we check that

$$r\otimes\mathrm{id}_B((a\otimes r)\otimes b) = ar\otimes b = a\otimes rb = \mathrm{id}_A\otimes l(a\otimes(r\otimes b)) = \mathrm{id}_A\otimes l(\alpha_{AkB}((a\otimes r)\otimes b)$$

showing the commutativity of the triangle diagram. □

1.3 Algebras in Vect_k

We now introduce k-algebras and algebras in the category of vector spaces and compare the classical definitions to the categorical definitions. In particular we show that these definitions are equivalent.

Definition 1.6. An *algebra* A over a field k is a vector space $A \in \mathrm{Vect}_k$ equipped with a bilinear product $A\times A\to A$, mapping $(x,y)\mapsto xy$. Additionally for all $x,y,z\in A$ we require the that multiplication is associative i.e. $(xy)z = x(yz)$ and there exist an element $1\in A$ such that $1x = x1 = x$. A is said to be an commutative k-algebra if for all $x,y\in A$ it holds that $xy = yx$.
An *algebra homomorphism* between k-algebras A and B is a linear map $f : A\to B$ such that for $1,x,y\in A$ and $1\in B$ the following equalities hold

$$\begin{aligned} f(1) &= 1 \\ f(xy) &= f(x)f(y). \end{aligned}$$

Using the previously defined tensor product of vector spaces we can define the notion of an algebra in the category of vector spaces over a field k.

Definition 1.7. An object $A\in\mathrm{Vect}_k$ is an *algebra object* in Vect_k if it can be equipped with maps $e : k\to A$ and $\mu : A\otimes A\to A$ such that the following diagrams commute

$$\begin{array}{ccccc} (A\otimes A)\otimes A & \xrightarrow{\alpha_{A,A,A}} & A\otimes(A\otimes A) & \xrightarrow{\mathrm{id}_A\otimes\mu} & A\otimes A \\ \downarrow{\scriptstyle \mu\otimes\mathrm{id}_A} & & & & \downarrow{\scriptstyle \mu} \\ A\otimes A & & \xrightarrow{\quad\mu\quad} & & A \end{array} \qquad \begin{array}{ccccc} k\otimes A & \xrightarrow{e\otimes\mathrm{id}_A} & A\otimes A & \xleftarrow{\mathrm{id}_A\otimes e} & A\otimes k \\ & {\scriptstyle l}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle r} & \\ & & A & & \end{array}$$

Here l and r denote the left and right unitors. We will call the first diagram the associativity axiom and the second diagram the unitality axiom. The triple (A,μ,e) is called an algebra in Vect_k. An algebra A is called commutative if

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\tau_{A,A}} & A\otimes A \\ & \searrow^{\mu} \quad {}^{\mu}\swarrow & \\ & A & \end{array}$$

commutes. We will call this the commutativity axiom. Here $\tau_{A,A}$ denotes the map $x\otimes y \mapsto y\otimes x$. An homomorphism of algebras A, B is a map $f \in \mathrm{Vect}_k(A,B)$ such that the following two diagram commutes

$$\begin{array}{ccc} A\otimes A & \xrightarrow{f\otimes f} & B\otimes B \\ \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \mu_B} \\ A & \xrightarrow{f} & B \end{array} .$$

$$\begin{array}{ccc} k & \xrightarrow{e_1} & A \\ & {}_{e_2}\searrow & \downarrow{\scriptstyle f} \\ & & B \end{array}$$

We now prove our first theorem, namely that these notions of algebra are equivalent.

Theorem 1.8. *If $A \in \mathrm{Vect}_k$ then*

1. *A is an k-algebra if and only if A is an algebra object in Vect_k.*
2. *A is an commutative k-algebra if and only if A is an commutative algebra object in Vect_k.*
3. *$f : A \to B$ is an homomorphism of k-algebras if and only if $f : A \to B$ is an homomorphism of algebras A and B.*

Proof. 1. Let A be a k-algebra. Then the composition of A induces a linear map $\mu : A\otimes A \to A$ induced by the universal property of the tensor product. We know check that the coherence axioms are satisfied. Note that it is adequate to see this is satisfied for pure tensors. Thus let $x, y, z \in A$. Then

$$(\mu(\mu\otimes \mathrm{id}_A))((x\otimes y)\otimes z) = \mu((xy)\otimes z) = (xy)z = x(yz).$$

Here the last equality is the associativity of the product in A. Now similary

$$(\mu(\mathrm{id}_A \otimes \mu(\alpha_{A,A,A})))((x\otimes y)\otimes z) = (\mu(\mathrm{id}_A \otimes \mu))(x\otimes (y\otimes z)) = \mu(x\otimes (yz)) = x(yz)$$

showing that the associativity diagram commutes. Furthermore let $r \in k$. Then we define $e : k \to A$ to be the linear map defined by $k \ni 1 \mapsto 1 \in A$. Then

$$(\mu(e\otimes \mathrm{id}_A))(r\otimes x) = \mu((r\cdot 1)\otimes x) = \mu(1\otimes (r\cdot x)) = 1(r\cdot x) = r\cdot x = l(r\otimes x)$$

showing the commutativity with the left unitor. The proof for the right unitor is analogous and therefor omitted. This shows that A is an algebra object.
Now let (A, μ, e) be an algebra object. Then define the product to be the composite $A\times A \xrightarrow{p} A\otimes A \xrightarrow{\mu} A$. Here p denotes the bilinear map defined by $(x,y) \mapsto x\otimes y$. We now show associativity and unitality. Let $x, y, z \in A$. Then

$$(xy)z = \mu(p(\mu(p(x,y)), z)) = \mu(\mu(x\otimes y)\otimes z) \stackrel{*}{=} \mu(x\otimes \mu(y\otimes z)) = \mu(p(x, \mu(p(y,z)))) = x(yz)$$

Here "*" follows from the associativity diagram showing that the multiplication is associative. Now we show that $1 := e(1)$ acts as the unit in in A. Let $x \in A$. Then by the unitality diagram

$$1x = e(1)x = (\mu(e\otimes \mathrm{id}_A))(1\otimes x) = l(1\otimes x) = x = r(x\otimes 1) = (\mu(\mathrm{id}_A \otimes e))(x\otimes 1) = xe(1) = x1$$

the bilinearity of the multiplication and the construction of the tensor product ensures that scaler multiplication satisfies the appropriate relations. This shows that A is an k-algebra completing the proof of 1.

2. Let A be an commutative k-algebra. By 1. A is an algebra object. Thus for $x, y \in A$ and using the commutativity of A it holds that $\mu(x \otimes y) = xy = yx = \mu(\tau_{A,A}(x \otimes y))$. Similary if A is an commutative algebra then by 1. A is an k-algebra. Thus since the commutativity diagram commutes $xy = \mu(x \otimes y) = \mu(y \otimes x) = yx$. This shows that A is an commutative k-algebra.

3. Let A, B be k-algebras and $f : A \to B$ an homomorphism of k-algebras. If $x, y \in A$ then

$$f(\mu_A(x \otimes y)) = f(xy) = f(x)f(y) = \mu_B(f(x) \otimes f(y)) = (\mu_B(f \otimes f))(x \otimes y).$$

   Similary if $f : A \to B$ is a homomorphism of algebras then in particular f is linear. Also for $x, y \in A$ it follows that

$$f(xy) = f(\mu_A(p(x, y))) = \mu_B((f \otimes f)(p(x, y))) = f(x)f(y).$$

   evaluating at $1 \in A$ shows that f preserves units thus finishing the proof.

□

From now on these notions of algebras will be used interchangeably.

Example 1.9. If G is a group one can define an k-algebra $k[G] = \mathrm{span}(G)$ where for $a, b \in k$ and $g, h \in G$ we define the multiplication by $(ag)(bh) = abgh$, extending with distributive laws, and with unit $e \in k[G]$.

1.4 Modules over algebras

In this section we will define the notion of a module over an algebra and homomorphisms between such modules. Furthermore we show that representations of a group G corresponds bijectively to modules over the group algebra $k[G]$. In fact this extends to an equivalence $\mathrm{Rep}(G) \simeq \mathbb{C}[G]\text{-Mod}(\mathrm{Vect}_{\mathbb{C}})$ of the category of k-linear representations of G with the category of $k[G]$-modules.

Definition 1.10. If A is an algebra over a field k a *left module* over A is a vector space $N \in \mathrm{Vect}_k$ equipped with a map $\rho : A \otimes N \to N$ such that the following diagrams commute

$$\begin{array}{ccc} k \otimes N & \xrightarrow{e \otimes \mathrm{id}_N} & A \otimes N \\ & {}_{l}\searrow & \downarrow{\scriptstyle \rho} \\ & & N \end{array} .$$

called the unitality diagram and the second diagram called the action property

$$\begin{array}{ccccc} (A \otimes A) \otimes N & \xrightarrow{\alpha_{AAN}} & A \otimes (A \otimes N) & \xrightarrow{\mathrm{id}_A \otimes \rho} & A \otimes N \\ \downarrow{\scriptstyle \mu \otimes \mathrm{id}_N} & & & & \downarrow{\scriptstyle \rho} \\ A \otimes N & & \xrightarrow{\qquad\qquad \rho \qquad\qquad} & & N \end{array}$$

A homomorphism $f : (N_1, \rho_1) \to (N_2, \rho_2)$ of left A modules is a linear map $f : N_1 \to N_2$ such that

$$\begin{array}{ccc} A \otimes N_1 & \xrightarrow{\mathrm{id}_A \otimes f} & A \otimes N_2 \\ \rho_1 \downarrow & & \downarrow \rho_2 \\ N_1 & \xrightarrow[f]{} & N_2 \end{array}$$

commutes.

Remark. The category of left modules over an k-algebra A is usually denoted A-Mod(Vect_k).

A reoccurring construction in this project is category of representations of a group G.

Definition 1.11. Let G be a group. A *representation* (V, ρ) of G is a vector space V over $\mathbb{C}$ and a group homomorphism $\rho : G \to \mathrm{Aut}(V)$. [1]

Remark. For a representation (V, ρ) of G we define the *dimension of the representation* to be $\dim(V, \rho) = \dim V$.

Definition 1.12. If (V, ρ_V) and (W, ρ_W) are representations of a group G, a linear map $f : V \to W$ is *G-linear* if for all $g \in G$

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \rho_V(g) \downarrow & & \downarrow \rho_W(g) \\ V & \xrightarrow[f]{} & W \end{array}$$

commutes.

It is clear that the identity map is G-linear and that the composition of G-linear maps is again G-linear. We now have the ingredients to define the category of representations of a group G.

Definition 1.13. For a group G. The category $\mathrm{Rep}\, G$ of representations of a group G, has as objects finite dimensional representations (V, ρ_V) and as arrows G-linear maps.

We will now conclude this chapter by showing our main theorem

Theorem 1.14. *If G is a group then* $\mathbb{C}[G]\text{-}\mathrm{Mod}(\mathrm{Vect}_{\mathbb{C}}) \simeq \mathrm{Rep}\, G$

Proof. If V is a left $C[G]$-module then the composite

$$G \xrightarrow{i} \mathbb{C}[G] \xrightarrow{\rho(-\otimes -)} \mathrm{Aut}\, V$$

With $g \mapsto \rho(g \otimes =)$. This is clearly well defined, since G is a group. It is also easily seen to be a group homomorphism. If on the other hand (V, ρ) is a representation The map $p \colon \mathbb{C}[G] \otimes V \to V$ with $g \otimes v \mapsto \rho_V(g)(v)$, defines an action on V. □

2 Monoidal categories

We have now discussed the notion of algebras in the category Vect_k, first through a classical description and then through a more modern categorical approach and shown that these are in fact equivalent notions. We have shown that Vect_k has a lot of additional structure. The goal of this chapter will be define the categorification of this structure (monoidal categories) and describe additional examples of such categories.

Definition 2.1. A monoidal category consists of the following data

[1] We will mainly concern us with complex representations in this project and therefor this definition is specified further, it is however easy to generalise.

- A category $\mathscr{C}$.
- A bifunctor
$$\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}.$$
- An object $1 \in \mathscr{C}$ called the unit.
- A natural isomorphism $\alpha : (- \otimes -) \otimes - \Rightarrow - \otimes (- \otimes -)$, such that for all $A, B, C, D \in \mathscr{C}$

$$\begin{array}{ccccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{ABC} \otimes \mathrm{id}_D} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{AB\otimes CD}} & A \otimes ((B \otimes C) \otimes D) \\
\downarrow{\scriptstyle \alpha_{A\otimes BCD}} & & & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \alpha_{BCD}} \\
(A \otimes B) \otimes (C \otimes D) & & \xrightarrow{\qquad\qquad \alpha_{ABC\otimes D} \qquad\qquad} & & A \otimes (B \otimes (C \otimes D))
\end{array}$$

  commutes.
- And two natural isomorphisms $r : - \otimes 1 \Rightarrow -$ and $l : 1 \otimes - \to -$ such that

$$\begin{array}{ccc}
(A \otimes 1) \otimes B & \xrightarrow{\alpha_{AkB}} & A \otimes (1 \otimes B) \\
{\scriptstyle r\otimes \mathrm{id}_B} \searrow & & \swarrow {\scriptstyle \mathrm{id}_A \otimes l} \\
 & A \otimes B &
\end{array}$$

  commutes for all $A, B \in \mathscr{C}$.

Notation. Given a monoidal category $(\mathscr{C}, \otimes, 1, \alpha, l, r)$ we will usually suppress, the associators and unitors and write $(\mathscr{C}, \otimes, 1)$.

Example 2.2. In fact Theorem 2.5 shows that $(\mathrm{Vect}_k, \otimes, k)$ is indeed a monoidal category. Another example of a monoidal category is the monoidal category $(\mathrm{Set}, \times, \{*\})$, with Set as category, the cartesian product and the one point set as unit. Additionally For any monoidal category $(\mathscr{C}, \otimes, 1)$ the category $(\mathscr{C}^{op}, \otimes_{op}, 1)$ with $A \otimes_{op} B :- B \otimes A$ for all $A, B \in \mathscr{C}^{op}$.

We will now use the monoidal structure on $\mathrm{Vect}_{\mathbb{C}}$ to produce a monoidal structure on $\mathrm{Rep}\, G$.

Definition 2.3. Let (V, ρ_v) and (W, ρ_W) be representations of a group G then we define the the *tensor product of the representations* (V, ρ_V) and (W, ρ_W) to be the representation $(V \otimes W, \rho_{V\otimes W})$ with $\rho_{V\otimes W}(g)(-) := \rho_V(g)(-) \otimes \rho_W(g)(-)$ for all $g \in G$. For G-linear maps $f : X \to Y$ and $g : X' \to Y'$ we define the tensor product of G-linear maps to be the tensor product linear maps $f \otimes g : X \otimes X' \to Y \otimes Y'$.

It is clear that the tensor product of representations is again a representation we will now show that the tensor product of G-linear maps is indeed G-linear.

Notation. From now on we will suppress the group homomorphisms and just say that V is a representation is a representation of a group G. Thus from now on ρ_V will always mean the corresponding group homomorphism $\rho_V : G \to \mathrm{Aut}\, V$.

Proposition 2.4. *If X, X', Y and Y' are representations of a group G and $f : X \to X'$ and $h : Y \to Y'$ are G-linear maps then $f \otimes h : X \otimes Y \to X' \otimes Y'$ is G-linear.*

Proof. Let $g \in G$ then for all $x \otimes y \in X \otimes Y$

$$\begin{aligned}
f \otimes h(\rho_{X\otimes Y}(g)(x \otimes y)) &= f \otimes h(\rho_X(g)(x) \otimes \rho_Y(g)(y)) = f(\rho_X(g)(x)) \otimes h(\rho_Y(g)) \\
&\overset{*}{=} \rho_{X'}(g)(f(x)) \otimes \rho_{Y'}(g)(h(y)) = \rho_{X'\otimes Y'}(g)(f(x) \otimes h(y)) = \rho_{X'\otimes Y'}(g)(f \otimes h(x \otimes y))
\end{aligned}$$

at "*" we use the G-linearity of f and h thus completing the proof. □

Corrolary. *For all $V, W, U \in \operatorname{Rep} G$ the associator $\alpha_{VWU} : (V \otimes W) \otimes U \to V \otimes (W \otimes U)$ is G-linear and natural in all arguments.*

This is clear from the definitions and the proof therefor is omitted.

Proposition 2.5. *For a group G the category $\operatorname{Rep} G$ is monoidal with the tensor product of representations and unit $\mathbb{C}$ given the trivial representation $t : G \to \operatorname{Aut} \mathbb{C}$ defined $g \mapsto \mathrm{id}$ for all $g \in G$.*

Proof. We now check that the left and right unitors are G-linear. If V is a representation of G then for all $g \in G$ and all $r \otimes v \in \mathbb{C} \otimes V$

$$l(t(g)(r) \otimes \rho_V(g)(v)) = l(1 \otimes r\rho_V(g)(v)) = \rho_V(g)(rv) = \rho_V(g)(l(r \otimes v).$$

The proof for the right unitor is analogous and therefor omitted then by Theorem 2.5 the pentagon and triangle diagram commutes, thus showing $(\operatorname{Rep} G, \otimes, (\mathbb{C}, t))$ is a monoidal category. ☐

This gives us our second example of a monoidal category.

Remark. A monoidal category $(\mathscr{C}, \otimes, 1)$ is called strict if the associator, the left and right unitors are all identity maps. In fact a theorem due to Mac Lane [Mac13] states that every monoidal category is equivalent to a strict monoidal category.

In an attempt to abstract from the notions algebras defined in definition 2.7 one defines the following.

Definition 2.6. If $(\mathscr{C}, \otimes, 1)$ is a monoidal category a *monoid object* in $\mathscr{C}$ consists of the following

- an object $A \in \mathscr{C}$.
- A map $e : 1 \to A$, this is usually referred to as the unit map.
- A map $\mu : A \otimes A \to A$, usually referred to as the multiplication map.

Such that the following diagrams (associativity)

$$\begin{array}{ccccc}
(A \otimes A) \otimes A & \xrightarrow{\alpha_{AAA}} & A \otimes (A \otimes A) & \xrightarrow{\mathrm{id}_A \otimes \mu} & A \otimes A \\
\downarrow{\scriptstyle \mu \otimes \mathrm{id}_A} & & & & \downarrow{\scriptstyle \mu} \\
A \otimes A & & \xrightarrow[\mu]{\hspace{6em}} & & A
\end{array}$$

and (unitality)

$$\begin{array}{ccccc}
1 \otimes A & \xrightarrow{e \otimes \mathrm{id}_A} & A \otimes A & \xleftarrow{\mathrm{id}_A \otimes e} & A \otimes 1 \\
& {\scriptstyle l} \searrow & \downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle r} & \\
& & A & &
\end{array}$$

commute.

A homomorphism of monoids (A, μ_A, e_A) and (B, μ_B, e_B) is a map $f : A \to B$ such that the following diagrams

$$\begin{array}{ccc}
A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\
\downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \mu_B} \\
A & \xrightarrow[f]{} & B
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
1 & \xrightarrow{e_A} & A \\
& {\scriptstyle e_B} \searrow & \downarrow{\scriptstyle f} \\
& & B
\end{array}$$

commute.

Remark. We will denote the category of monoids in a monoidal category $\mathscr{C}$ by $\operatorname{Mon}\mathscr{C}$. Thus part 1 and 3 of theorem 2.8 establishes that $\operatorname{Mon}(\operatorname{Vect}_k) \simeq \operatorname{Alg}_k$. Where Alg_k denotes the category of k-algebras and algebra homomorphisms.

In a similar fashion we can define the notion of a module over a monoid.

Definition 2.7. If (A, μ, e) is a monoid in a monoidal category $\mathscr{C}$ a left module over A consists of

- an object $N \in \mathscr{C}$.
- A map $\rho : A \otimes N \to N$ called the action.

Such that

1. (Unitality) The following diagram commutes

$$\begin{array}{ccc} k \otimes N & \xrightarrow{e \otimes \mathrm{id}_N} & A \otimes N \\ & \searrow^{l} & \downarrow \rho \\ & & N \end{array} .$$

2. (action property) and the following diagram

$$\begin{array}{ccccc} (A \otimes A) \otimes N & \xrightarrow{\alpha_{AAN}} & A \otimes (A \otimes N) & \xrightarrow{\mathrm{id}_A \otimes \rho} & A \otimes N \\ \downarrow \mu \otimes \mathrm{id}_N & & & & \downarrow \rho \\ A \otimes N & & \xrightarrow{\quad\quad\rho\quad\quad} & & N \end{array}$$

   commutes.

A homomorphism of A-modules (N, ρ_N) and (N, ρ_M) is a map $f : N \to M$ such that

$$\begin{array}{ccc} A \otimes N & \xrightarrow{\mathrm{id}_A \otimes f} & A \otimes M \\ \rho_N \downarrow & & \downarrow \rho_M \\ N & \xrightarrow[f]{} & M \end{array}$$

commutes. We denote the category of left A-modules in $\mathscr{C}$ by $A\operatorname{Mod}\text{-}\mathscr{C}$.

This concludes the first step in the process of defining symmetric fusion categories which Tannaka duality concerns.

3 Monoidal functors

We will in this section explore notion of structure preserving morphisms between monoidal and k-linear categories and maps between these. Additionally we will prove that monoidal functors preserve duals and use this fact to prove that given two monoidal categories out of a rigid category any monoidal natural transformation between these will be an monoidal isomorphism.

Definition 3.1. Let $(\mathscr{C}, \otimes_{\mathscr{C}}, 1_{\mathscr{C}})$ and $(\mathscr{D}, \otimes_{\mathscr{D}}, 1_{\mathscr{D}})$ be monoidal categories. A *lax monoidal functor* is a

- functor $F\colon \mathscr{C} \to \mathscr{D}$.
- A morphism $\varepsilon\colon 1_{\mathscr{D}} \to F(1_{\mathscr{C}})$.
- A natural transformation with components $\mu_{X,Y}\colon F(X) \otimes_{\mathscr{D}} F(Y) \to F(X \otimes_{\mathscr{C}} Y)$ for all $X, Y \in \mathscr{C}$.

Such that for all $X, Y, Z \in \mathscr{C}$ the following diagrams commute:

- (Associativity)

$$\begin{array}{ccc}
(F(X)\otimes_{\mathscr{D}} F(Y))\otimes_{\mathscr{D}} F(Z) & \xrightarrow{\quad\alpha\quad} & F(X)\otimes_{\mathscr{D}}(F(Y)\otimes_{\mathscr{D}} F(Z)) \\
{\scriptstyle \mu_{X,Y}\otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\mu_{Y,Z}} \\
F(X\otimes_{\mathscr{C}} Y)\otimes_{\mathscr{D}} F(Z) & & F(X)\otimes_{\mathscr{D}}(F(Y\otimes_{\mathscr{C}} Z)) \\
\downarrow{\scriptstyle \mu_{X\otimes Y,Z}} & & \downarrow{\scriptstyle \mu_{X,Y\otimes Z}} \\
F((X\otimes_{\mathscr{C}} Y)\otimes_{\mathscr{C}}) & \xrightarrow[F(\alpha)]{\qquad} & F(X\otimes_{\mathscr{C}}(Y\otimes_{\mathscr{C}} Z))
\end{array} .$$

- (Unitality)

$$\begin{array}{ccc}
1_{\mathscr{D}}\otimes_{\mathscr{D}} F(X) & \xrightarrow{\varepsilon\otimes\mathrm{id}} & F(1_{\mathscr{C}})\otimes_{\mathscr{D}} F(X) \\
\downarrow{\scriptstyle l_{\mathscr{D}}} & & \downarrow{\scriptstyle \mu_{1_{\mathscr{C}},X}} \\
F(X) & \xleftarrow[F(l_{\mathscr{C}})]{\qquad} & F(1\otimes_{\mathscr{C}} X)
\end{array}$$

and

$$\begin{array}{ccc}
F(X)\otimes_{\mathscr{D}} 1_{\mathscr{D}} & \xrightarrow{\mathrm{id}\otimes\varepsilon} & F(X)\otimes_{\mathscr{D}} F(1_{\mathscr{C}}) \\
\downarrow{\scriptstyle r_{\mathscr{D}}} & & \downarrow{\scriptstyle \mu_{X,1_{\mathscr{C}}}} \\
F(X) & \xleftarrow[F(r_{\mathscr{C}})]{\qquad} & F(X\otimes_{\mathscr{C}} 1_{\mathscr{C}})
\end{array} .$$

Monoidal functors will play the role of structure preserving functors between monoidal categories. One example of this is the following proposition:

Proposition 3.2. *Let $(\mathscr{C}, \otimes_{\mathscr{C}}, 1_{\mathscr{C}})$ and $(\mathscr{D}, \otimes_{\mathscr{D}}, 1_{\mathscr{D}})$ be monoidal categories and $F\colon \mathscr{C} \to \mathscr{D}$ a lax monoidal functor. If (A, μ, e) is a monoid object in $\mathscr{C}$ then $F(A)$ can be made into monoid in $\mathscr{D}$ with multiplication given by the composite*

$$\mu^*\colon F(A)\otimes_{\mathscr{D}} F(A) \xrightarrow{\eta_{A,A}} F(A\otimes_{\mathscr{C}} A) \xrightarrow{F(\mu)} F(A)$$

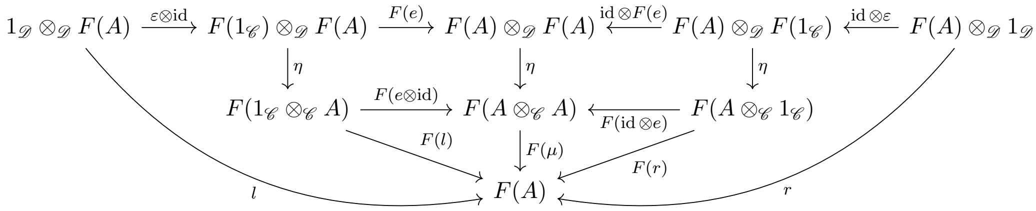
and unit map given by

$$e^*\colon 1_{\mathscr{D}} \xrightarrow{\quad\varepsilon\quad} F(1_{\mathscr{C}}) \xrightarrow{F(e)} F(A) \ .$$

Proof. Let $\eta\colon F(-)\otimes_{\mathscr{D}} F(-) \Rightarrow F(-\otimes_{\mathscr{C}} -)$ be a the natural transformation F comes equipped with. We consider the diagram

$$\begin{tikzcd}
(F(A)\otimes_{\mathscr{D}} F(A))\otimes_{\mathscr{D}} F(A) \arrow[r, "\alpha"] \arrow[d, "\eta\otimes\mathrm{id}"'] & F(A)\otimes_{\mathscr{D}}(F(A)\otimes_{\mathscr{D}} F(A)) \arrow[rr, "\mathrm{id}\otimes\eta"] & & F(A)\otimes_{\mathscr{D}} F(A\otimes_{\mathscr{C}} A) \arrow[dl, "\eta"'] \arrow[d, "\mathrm{id}\otimes F(\mu)"] \\
F(A\otimes_{\mathscr{C}} A)\otimes_{\mathscr{D}} F(A) \arrow[r, "\eta"] \arrow[d, "F(\mu)\otimes\mathrm{id}"'] & F((A\otimes_{\mathscr{C}} A)\otimes_{\mathscr{C}} A) \arrow[r, "F(\alpha)"] \arrow[ddl, "F(\mu\otimes\mathrm{id})"] & F(A\otimes_{\mathscr{C}}(A\otimes_{\mathscr{C}} A)) \arrow[dr, "F(\mathrm{id}\otimes\mu)"] & F(A)\otimes_{\mathscr{D}} F(A) \arrow[d, "\eta"] \\
F(A)\otimes_{\mathscr{D}} F(A) \arrow[d, "\eta"'] & & & F(A\otimes_{\mathscr{C}} A) \arrow[d, "F(\mu)"] \\
F(A\otimes_{\mathscr{C}} A) \arrow[rrr, "F(\mu)"'] & & & F(A)
\end{tikzcd}$$

by functorality of $\otimes_{\mathscr{D}}$ the outer square of the diagram is exactly the associativity coherence of the multiplication of $F(A)$. Since A is monoid in $\mathscr{C}$ the functorality of F makes the lower pentagon commutes. The two triangles commute by the naturality of η and the upper square commutes by the monoidality of F. Hence μ^* satisfies the associativity axiom. In a similar sense we consider the diagram



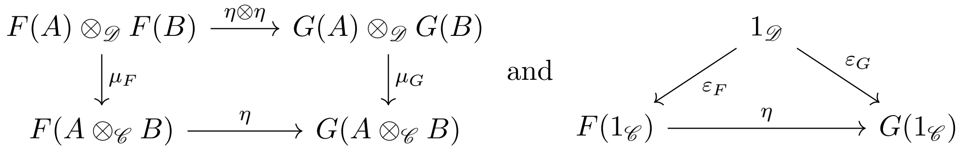
. The commutativity of this diagram ensures that e^* satisfies the unitality condition in a similar fashions as the commutativity of the former diagram assured associativity. For the commutativity of this diagram we only show the commutativity the left side, since the the argument showing the commutativity of the right side is analogous. The cell to the left is commutative by the unitality condition of F. Since A is a monoid in $\mathscr{C}$ the functorality of F ensures the commutativity of the inner triangle. At last the naturality of η ensures the commutativity of the inner square. Hence $(F(A), \mu^*, e^*)$ is a monoid in $\mathscr{D}$. $\square$

Definition 3.3. Let $(\mathscr{C}, \otimes_{\mathscr{C}}, 1_{\mathscr{C}})$ and $(\mathscr{D}, \otimes_{\mathscr{D}}, 1_{\mathscr{D}})$ be monoidal categories a lax monoidal functor $F\colon \mathscr{C} \to \mathscr{D}$ is a *monoidal functor* $F\colon \mathscr{C} \to \mathscr{D}$ if the map ε is an isomorphism and the natural transformation μ is an natural isomorphism.

Example 3.4. The forgetful functor $U\colon \operatorname{Rep} G \to \mathrm{Vect}_{\mathbb{C}}$ is a monoidal functor.

And this extends to the notion of natural transformations aswell.

Definition 3.5. Let $(\mathscr{C}, \otimes_{\mathscr{C}}, 1_{\mathscr{C}})$ and $(\mathscr{D}, \otimes_{\mathscr{D}}, 1_{\mathscr{D}})$ be monoidal categories and $F, G\colon \mathscr{C} \rightrightarrows \mathscr{D}$ be monoidal functors. A natural transformation $\eta\colon F \Rightarrow G$ is a *monoidal natural transformation* if for all $A, B \in \mathscr{C}$ the following diagrams



commute.

Definition 3.6. If $F\colon \mathscr{C} \to \mathscr{D}$ is a monoidal functor then we define the group of monoidal automorphisms on F to be the group $\mathrm{Aut}^{\otimes} F$. With composition of natural transformations to as multiplication.

4 Braiding and symmetry

In this section we will define the notion of a braided monoidal category and extend this to a symmetric monoidal category. Furthermore we will show that Vect_k is a symmetric monoidal category and extend this to $\operatorname{Rep} G$ for a group G.

Definition 4.1. Let $(\mathscr{C}, \otimes, 1)$ be a monoidal category and $S\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ the functor defined by $S(A, B) = (B, A)$ and similary for maps. A *braiding* β on $\otimes$ is a monoidal natural isomorphism depicted in the diagram below

$$\begin{array}{ccc} \mathscr{C}\times\mathscr{C} & \xrightarrow{\quad\otimes\quad} & \mathscr{C} \\ & {\scriptstyle S}\searrow \quad \Downarrow{\scriptstyle\beta} \quad \nearrow{\scriptstyle\otimes} & \\ & \mathscr{C}\times\mathscr{C} & \end{array}$$

Such that for all $A, B, C \in \mathscr{C}$ the following diagrams commute

$$\begin{array}{ccccc} & A\otimes(B\otimes C) & \xrightarrow{\beta} & (B\otimes C)\otimes A & \\ {}^{\alpha}\nearrow & & & & \searrow^{\alpha} \\ (A\otimes B)\otimes C & & & & B\otimes(C\otimes A) \\ {}_{\beta\otimes\mathrm{id}}\searrow & & & & \nearrow_{\mathrm{id}\otimes\beta} \\ & (B\otimes A)\otimes C & \xrightarrow[\alpha]{} & B\otimes(A\otimes C) & \end{array}$$

$$\begin{array}{ccccc} & (A\otimes B)\otimes C & \xrightarrow{\beta} & C\otimes(A\otimes B) & \\ {}^{\alpha^{-1}}\nearrow & & & & \searrow^{\alpha^{-1}} \\ A\otimes(B\otimes C) & & & & (C\otimes A)\otimes B \\ {}_{\mathrm{id}\otimes\beta}\searrow & & & & \nearrow_{\beta\otimes\mathrm{id}} \\ & A\otimes(C\otimes B) & \xrightarrow[\alpha^{-1}]{} & B\otimes(A\otimes C). & \end{array}$$

A monoidal category is called braided if it has a braiding. If for all $A, B \in \mathscr{C}$ $\beta_{B,A}\beta_{A,B} = \mathrm{id}_{A\otimes B}$ we say $\mathscr{C}$ is a *symmetric* monoidal category.

Remark. For a braided monoidal category $(\mathscr{C}, \otimes, 1, \beta)$ we will usually suppress the braiding and write $(\mathscr{C}, \otimes, 1)$ is a braided monoidal category.

Proposition 4.2. *For $V, W \in \mathrm{Vect}_k$ the map $\beta_{V,W}\colon V\otimes W \to W\otimes V$ defined by $v\otimes w \mapsto w\otimes v$ is a linear map additionally $\beta_{V,W}$ is an isomorphism natural in V and W and $\beta_{V,W}$ defines a symmetric braiding for $\otimes$.*

Proof. By the universal property of the tensor product it is clear that $\beta_{V,W}$ is a linear isomorphism of vector spaces. To see that $\beta_{V,W}$ is natural in V and W let $f\colon V \to V'$ and $g\colon W \to W'$. Then for $v\otimes w \in V\otimes W$

$$\beta_{V',W'}(f\otimes g)(v\otimes w) = g(w)\otimes f(v) = (g\otimes f)\beta_{V,W}(v\otimes w)$$

showing the naturality thus showing β is a natural isomorphism $\beta\colon \otimes \Rightarrow \otimes\circ S$. We will only show commutativity of the first diagram, since the argument for the 2nd is analogous. Let $U \in \mathrm{Vect}_k$ and $(v\otimes w)\otimes u \in (V\otimes W)\otimes U$ then

$$\alpha\beta\alpha\left((v\otimes w)\otimes u\right) = w\otimes(u\otimes v) = \mathrm{id}\otimes\beta\left(w\otimes(u\otimes v)\right) = (\mathrm{id}\otimes\beta)\alpha(\beta\otimes\mathrm{id})\left((v\otimes w)\otimes u\right)$$

showing the commutativity. The symmetry is clear from the definition of β. □

To see that this extends to $\mathrm{Rep}\,G$ we prove the following proposition:

Proposition 4.3. *If G be a group and $V, W \in \mathrm{Rep}\,G$ then $\beta_{V,W}$ is G-linear.*

Proof. It is clear that for $g \in G$

$$\beta_{V,W}(\rho_V(g)(v) \otimes \rho_W(g)(w)) = \rho_W(g)(w) \otimes \rho_V(g)(v) = \rho_W \otimes \rho_V(g)(\beta_{V,W}(v \otimes w))$$

thus $\beta_{V,W}$ is G-linear. □

This shows that $\operatorname{Rep} G$ is a symmetric monoidal category. We will now look back at monoid objects and define *commutative* monoid object.

Definition 4.4. Let $(\mathscr{C}, \otimes, 1)$ be a symmetric monoidal category and $M \in \mathscr{C}$ a monoid object. We say that M is a commutative monoid if the following diagram commutes

$$\begin{array}{ccc} M \otimes M & \xrightarrow{\beta_{M,M}} & M \otimes M \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & M & \end{array} .$$

Remark. We denote the category of commutative monoids in a symmetric monoidal category $\mathscr{C}$ by $\operatorname{CMon}\mathscr{C}$. Thus Theorem 2.8 establishes equivalence of categories namely that $\operatorname{CMon}(\mathrm{Vect}_k) \simeq \mathrm{CAlg}_k$, where CAlg_k denotes the category of commutative k-algebras and algebra homomorphisms.

We will now generalize the hom-tensor adjunction of vector spaces to the setting of monoidal categories.

Definition 4.5. Let $(\mathscr{C}, \otimes, 1)$ be a symmetric monoidal category. $\mathscr{C}$ is *closed* if for all $A \in \mathscr{C}$ the functor

$$- \otimes A \colon \mathscr{C} \to \mathscr{C}$$

Has a right adjoint $[-, A] \colon \mathscr{C} \to \mathscr{C}$. We will name the object $[A, B]$ the *internal hom* from A to B.

Notation. We will abuse notation and denote the internal hom of in a closed monoidal category $\mathscr{C}$ by the $\mathscr{C}(A, B)$ for $A, B \in \mathscr{C}$. In cases where the internal and external hom can not be identified in a natural way, we will denote the internal hom by $\underline{\mathscr{C}}(A, B)$.

Proposition 4.6. *If G is a group then* $\operatorname{Rep} G$ *is a closed monoidal category.*

Proof. We have already shown that $\operatorname{Rep} G$ is a symmetric monoidal category. Now for representations $V, W \in \operatorname{Rep} G$. The representation $\hom(V, W)$ with the action on on $\hom(V, W)$ given by $\rho_{\hom(V,W)}(g)(f) = \rho_W(g)(f(\rho_V(g)^{-1}(-)))$. We know that we have an adjunction

$$- \otimes A \dashv \mathrm{Vect}_k(A, -).$$

We will now show that the components of the unit and counit of the hom-tensor adjunction are G-linear with respect to the representation just defined. Thus lifting the adjunction to $\operatorname{Rep} G$. Let $g \in G$ and $v \in V$.

$$\begin{aligned} \varepsilon_V(\rho_{\hom(V,W)}(g)(f) \otimes \rho_V(g)(v)) &= \varepsilon_V(\rho_W(g)(f(\rho_V(g)^{-1}(-))) \otimes \rho_V(g)(v)) \\ &= \rho_W(g)(f(\rho_V(g)^{-1}(\rho_V(g)(v)))) \\ &= \rho_W(g)(f(\rho_V(g^{-1}g)(v))) \\ &= \rho_W(g)(f(v)). \end{aligned}$$

One similarly shows that the counit is G-linear. □

At last we have a notion of monoidal natural transformations compatible with braidings

Definition 4.7. Let $(\mathscr{C}, \otimes_{\mathscr{C}}, 1_{\mathscr{C}})$ and $(\mathscr{D}, \otimes_{\mathscr{D}}, 1_{\mathscr{D}})$ be braided monoidal categories. A (lax) monoidal functor is a braided (lax) monoidal functor if for all $A, B \in \mathscr{C}$ the following diagram

$$\begin{array}{ccc} F(A) \otimes_{\mathscr{D}} F(B) & \xrightarrow{\beta} & F(B) \otimes_{\mathscr{D}} F(A) \\ \mu\downarrow & & \downarrow\mu \\ F(A \otimes_{\mathscr{C}} B) & \xrightarrow[F(\beta)]{} & F(B \otimes_{\mathscr{C}} A) \end{array}$$

commutes.

Remark. If the braided categories mentioned are symmetric, we will say that F is a symmetric monoidal functor.

5 Duals and rigidity

During this chapter we will define the notion of duals in a monoidal category.

Definition 5.1. Let $(\mathscr{C}, \otimes, 1)$ be a monoidal category and $A \in \mathscr{C}$. We say that A^* is a *right dual* of A if there exists map $\mathrm{ev}\colon A \otimes A^* \to 1$ and $\mathrm{coev}\colon 1 \to A^* \otimes A$ such that the composites

$$A \xrightarrow{l^{-1}} A \otimes 1 \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} A \otimes (A^* \otimes A) \xrightarrow{(\mathrm{ev} \otimes \mathrm{id}) \circ \alpha^{-1}} 1 \otimes A \xrightarrow{r} A$$

and

$$A^* \xrightarrow{r^{-1}} 1 \otimes A^* \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} (A^* \otimes A) \otimes A \xrightarrow{(\mathrm{id} \otimes \mathrm{ev}) \circ \alpha} A^* \otimes 1 \xrightarrow{l} A$$

equal the identities. We say that $\mathscr{C}$ is *right rigid* if all objects has a right dual.

Their is an analogous definition of a left dual.

Definition 5.2. Let $(\mathscr{C}, \otimes, 1)$ be a monoidal category and $A \in \mathscr{C}$. We say that ${}^*A$ is a left dual of A if there exists maps $\mathrm{ev}'\colon {}^*A \otimes A \to 1$ and $\mathrm{coev}'\colon 1 \to A \otimes {}^*A$ satisfying similarly relations. A category in which every object has a left dual is called left rigid. A category in which every object has a right and left dual is called *rigid*.

Remark. These equations are normally called the snake equations.

We will finish of this section by showing that $\mathrm{Rep}\, G$ is rigid and we will do this in parts firstly we will show that the full subcategory Vec_k of finite dimensional vector spaces and linear maps is rigid and then show that the evaluation and coevaluation maps in Vec_k are G-linear thus showing that $\mathrm{Rep}\, G$ is rigid.

Proposition 5.3. *The category* Vec_k *is right rigid.*

Proof. Let V be a finite dimensional vector space. Consider the dual vector space $\hom(V, k)$ and the maps $\mathrm{ev}\colon V \otimes \hom(V, k) \to k$ given by $v \otimes f \mapsto f(v)$ and given a basis $\{e_1, \dots, e_n\}$ of V and its corresponding dual basis $\{e^1, \dots, e^n\}$ of $\hom(V, k)$ where we define

$$\mathrm{coev}\colon k \to \hom(V, k) \otimes V$$
$$r \longmapsto r \sum_{i=1}^{n} e^i \otimes e_i$$

We will now check that the composites defined in definition of rigid categories is indeed the identity. Let $v = \sum_{i=1}^{n} a_i e_i \in V$

$$\begin{aligned} v \mapsto v \otimes 1 \mapsto v \otimes \left(\sum_{i=1}^{n} e^i \otimes e_i \right) \\ = v \otimes \left(\sum_{i=1}^{n} e^i \otimes \sum_{i=1}^{n} e_i \right) \\ \mapsto \left(\sum_{i=1}^{n} a_i e_i \otimes \sum_{i=1}^{n} e^i \right) \otimes \sum_{i=1}^{n} e_i \\ \mapsto \sum_{i=1}^{n} a_i \otimes \sum_{i=1}^{n} e_i \mapsto \sum_{i=1}^{n} a_i e_i = v \end{aligned}$$

Showing that the 1st snake equation is satisfied. The argument for the 2nd is similar and it is therefor excluded. At last since this argument could have been givin in a similar fashion for left duals we conclude that Vec_k is rigid. □

We will now define the dual representation of a representation V and show that the previously defined maps ev, coev are G-linear.

Proposition 5.4. *For a group G the category* $\operatorname{Rep} G$ *is rigid.*

Proof. Let (V, ρ_V) be a representation of G, we define the dual representation to be the pair $(\hom(V, k), \rho_{V^*})$ where for all $v \in V$ $\rho_{V^*}(g)(f(v)) = f(\rho_V(g^{-1})(v))$. If $g \in G$ then

$$\begin{aligned} \mathrm{ev}(\rho_{V \otimes V^*}(g)(v \otimes f)) &= \mathrm{ev}(\rho_V(g)(v) \otimes \rho_{V^*}(g)(f)) \\ &= \rho_{V^*}(g)(f(\rho(g)(v))) \\ &= f(\rho_V(g^{-1})(\rho_V(g)(v))) \\ &= f(\rho_V(g^{-1}g)(v)) \\ &= f(v) = t(g)(f(v)) \\ &= t(g)(\mathrm{ev}(v \otimes f)) \end{aligned}$$

Showing that the evaluation map is G-linear. Since it is sufficient to check for basis elements we get that for $1 \in k$

$$\begin{aligned} \rho^*(g)(\mathrm{coev}(1)) = \rho^*(g) \left(\sum_{i=1}^{n} e^i \otimes e_i \right) &= \sum_{i=1}^{n} e^i(\rho(g^{-1})(-)) \otimes \rho(g)(e_i) \\ &= \sum_{i=1}^{n} \left(\sum_{m=1}^{n} g_{mi}^{-1} e^n \right) \otimes \left(\sum_{j=1}^{n} g_{ij} e_j \right) \\ &= \sum_{i=1}^{n} \sum_{m=1}^{n} \sum_{j=1}^{n} g_{mi}^{-1} g_{ij} (e^m \otimes e_j) \\ &= \sum_{j=1}^{n} \sum_{m=1}^{n} (\delta_{mj} (e^m \otimes e_j)) \\ &= \sum_{i=1}^{n} e^i \otimes e_i. \end{aligned}$$

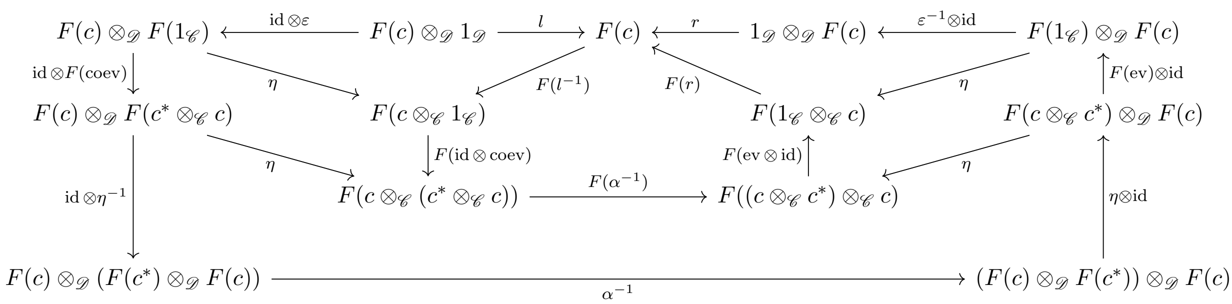
Thus showing that the coevaluation map is G-linear. Hence $\operatorname{Rep} G$ is a right rigid category. □

Theorem 5.5. *Let* $(\mathscr{C}, \otimes_{\mathscr{C}}, 1_{\mathscr{C}})$ *and* $(\mathscr{D}, \otimes_{\mathscr{D}}, 1_{\mathscr{D}})$ *be monoidal category and* $(F\colon \mathscr{C} \to \mathscr{D}, \eta, \varepsilon)$ *a monoidal functor. If* $c \in \mathscr{C}$ *has a right dual* $c^* \in \mathscr{C}$. *Then* $F(c^*)$ *is a right dual of* $F(c)$.

Proof. We claim that the composites

$$\mathrm{ev}^*\colon F(c) \otimes_{\mathscr{D}} F(c^*) \xrightarrow{\eta} F(c \otimes_{\mathscr{C}} c^*) \xrightarrow{F(\mathrm{ev})} F(1_{\mathscr{C}}) \xrightarrow{\varepsilon^{-1}} 1_{\mathscr{D}}$$
$$\mathrm{coev}^*\colon 1_{\mathscr{D}} \xrightarrow{\varepsilon} F(1_{\mathscr{C}}) \xrightarrow{F(\mathrm{coev})} F(c^* \otimes_{\mathscr{C}} c) \xrightarrow{\eta^{-1}} F(c^*) \otimes_{\mathscr{D}} F(c)$$

act as the evaluation and coevaluation maps of the pair $F(c)$ and $F(c^*)$. To prove that the first snake equation is satisfied it suffices to show that the following diagram commutes:

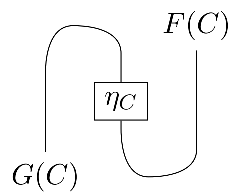


since the commutativity of the outer rim is equivalent to ev^* and coev^* satisfying the the first snake equation. The see that the diagram commutes one needs only realize that the triangles at the top commute by the unitality property of monoidal functors. The left and right square commute by the naturality of η. The inner pentagon commutes by since c and c^* satisfy the snake equation in and since F is a functor. At last the lower polygon commutes by the associativity constraint on F. The argument to see the second snake equation is satisfied is analogous and thus excluded. □

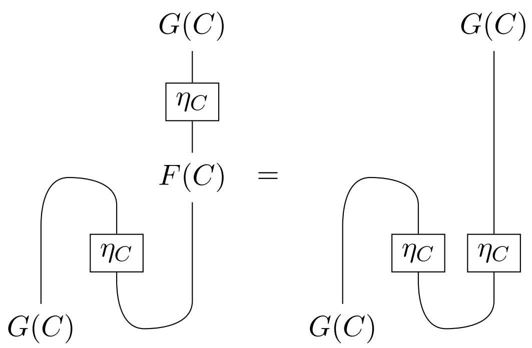
This is a very useful fact, which will be used not only in the next lemma, but it also allows us to prove a very important adjunction of functors.

Lemma 5.6. *Let* $(\mathscr{C}, \otimes_{\mathscr{C}}, 1_{\mathscr{C}})$ *and* $(\mathscr{D}, \otimes_{\mathscr{D}}, 1_{\mathscr{D}})$ *be monoidal categories with* $\mathscr{C}$ *rigid and* $F, G\colon \mathscr{C} \to \mathscr{D}$ *monoidal functors. If* $\eta\colon F \Rightarrow G$ *is a monoidal natural transformation then* η *is monoidal natural isomorphism.*

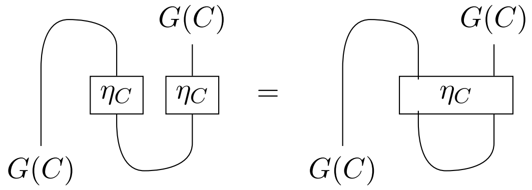
Proof. Let $F, G\colon \mathscr{C} \to \mathscr{D}$ monoidal functors from a rigid monoidal category $\mathscr{C}$ and $\eta\colon F \to G$ a monoidal natural transformation. Let $C \in \mathscr{C}$ and consider the map:



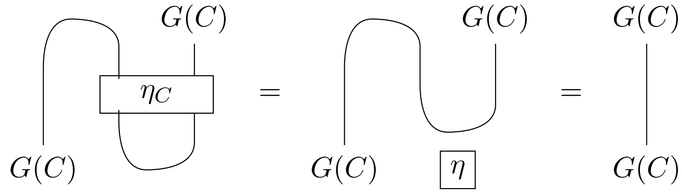
The claim is that this is an inverse to η_C. By functorality we conclude the following



Then by monoidality of η we conclude:



then by naturality of η



Showing that this is indeed a left inverse of η_C. The proof showing that this is also a right inverse is similar and thus omitted from this exposition. $\square$

6 Enriched categories and Abelian categories

In this section discuss enriched categories, a categorical construction enabling that the hom-sets in a category to have the additional structure of being objects in a monoidal category. Expanding further one this notion we will define abelian categories, which are categories similar to the category Ab of abelian groups and group homomorphisms. Finally this enables us define short exact sequences.

6.1 Enriched categories

We will now define enriched categories, functors and natural transformations.

Definition 6.1. Let $(\mathscr{V}, \otimes, 1)$ be a monoidal category. A category $\mathscr{C}$ *enriched* in $\mathscr{V}$ consists of the following:

- a collection of objects $\text{Ob}\,\mathscr{C}$.
- For all objects $A, B \in \mathscr{C}$ an object $C(A, B) \in \mathscr{V}$.
- For all maps $f \in \mathscr{C}(A, B)$ a map $f : 1 \to C(A, B)$ in $\mathscr{V}$.
- A map $\text{id}_A : 1 \to C(A, A)$ in $\mathscr{V}$ corresponding to the identity arrow $\text{id}_A : A \to A$ in $\mathscr{C}$.
- For all $A, B, C \in \mathscr{C}$ map $\circ_{ABC} : C(B, C) \otimes C(A, B) \to C(A, C)$ in $\mathscr{V}$.

Such that the following three diagrams commute

$$\begin{array}{ccc}
(C(C,D)\otimes C(B,C))\otimes C(A,B) & \xrightarrow{\circ_{BCD}\otimes \mathrm{id}_{C(A,B)}} & C(B,D)\otimes C(A,B) \\
 & & \downarrow{\scriptstyle \circ_{ABD}} \\
\downarrow{\scriptstyle \alpha} & & C(A,D) \\
 & & \uparrow{\scriptstyle \circ_{ACD}} \\
C(C,D)\otimes (C(B,C)\otimes C(A,B)) & \xrightarrow[\mathrm{id}_{C(C,D)}\otimes \circ_{ABC}]{} & C(C,D)\otimes C(A,C)
\end{array}$$

and

$$\begin{array}{ccc}
1\otimes C(A,B) & \xrightarrow{\mathrm{id}_B\otimes \mathrm{id}_{C(A,B)}} & C(B,B)\otimes C(A,B) \\
 & {\scriptstyle l}\searrow \quad \swarrow{\scriptstyle \circ_{ABB}} & \\
 & C(A,B) &
\end{array},$$

$$\begin{array}{ccc}
C(A,B)\otimes 1 & \xrightarrow{\mathrm{id}_{C(A,B)}\otimes \mathrm{id}_A} & C(A,B)\otimes C(A,A) \\
 & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle \circ_A AB} & \\
 & C(A,B) &
\end{array}$$

for all $A, B, C, D \in \mathscr{C}$.

Example 6.2. The category Vect_k of vector spaces of a field k is enriched over the category $(Vect_k, \otimes, k)$.For all pairs of vector spaces V, W the vector space $\mathrm{Vect}_k(V, W)$ to be the vector space of morphisms. The composition is constructed with the universal property of the tensor product. Showing that the appropriate diagrams commute is a matter of diagram chase, much similar to how we checked commutativity several times during this project.

Definition 6.3. Let $(\mathscr{V}, \otimes, 1)$ be a monoidal category and $\mathscr{C}, \mathscr{D}$ be $\mathscr{V}$-categories. A $\mathscr{V}$*-enriched functor* $F: \mathscr{C} \to \mathscr{D}$ ($\mathscr{V}$-functor for short) consists of the following:

- A map $F\colon \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$.
- For all $X, Y \in \mathscr{C}$ a map
$$F_{X,Y} : \mathscr{C}(X,Y) \to \mathscr{D}(F(X), F(Y))$$
in $\mathscr{V}$ such that the following diagrams commute for all $X, Y, Z \in \mathscr{C}$

$$\begin{array}{ccc}
\mathscr{C}(Y,Z)\otimes \mathscr{C}(X,Y) & \xrightarrow{\circ} & \mathscr{C}(X,Z) \\
{\scriptstyle F_{Y,Z}\otimes F_{X,Y}}\downarrow & & \downarrow{\scriptstyle F_{X,Z}} \\
\mathscr{D}(F(Y),F(Z))\otimes \mathscr{D}(F(X),F(Y)) & \xrightarrow[\circ]{} & \mathscr{D}(F(X),F(Z))
\end{array}
\qquad
\begin{array}{ccc}
1 & \xrightarrow{\mathrm{id}_X} & \mathscr{C}(X,X) \\
 & {\scriptstyle \mathrm{id}_{F(X)}}\searrow & \downarrow{\scriptstyle F_{X,X}} \\
 & & \mathscr{D}(F(X),F(X)).
\end{array}$$

additionally there is also a notion of enriched natural transformation.

Definition 6.4. Let $(\mathscr{V}, \otimes, 1)$ be a monoidal category, $\mathscr{C}, \mathscr{D}$ be $\mathscr{V}$-categories and $F, G: \mathscr{C} \to \mathscr{D}$ be $\mathscr{V}$-functors. A $\mathscr{V}$-natural transformation $\alpha\colon F \Rightarrow G$ consists of a collection of morphisms $\alpha_X : 1 \to \mathscr{D}(F(X), G(X))$ indexed by $X \in \mathscr{C}$. Such that for all $X, Y \in \mathscr{C}$ the following diagram commutes

$$\begin{array}{ccc}
\mathscr{C}(X,Y) & \xrightarrow{F_{X,Y}} & \mathscr{D}(F(X),F(Y)) \\
\downarrow{\scriptstyle G_{X,Y}} & & \downarrow{\scriptstyle (\alpha_Y)_*} \\
\mathscr{D}(G(X),G(Y)) & \xrightarrow{(\alpha_X)^*} & \mathscr{D}(F(X),G(X))
\end{array}$$

With this we have established all the theory of a general enriched category, that we will need.

6.2 Additive categories

We will now specialize the theory to categories and enriched in abelian groups. Furthermore we will look at additive categories which are Ab-categories with additional structure. This will lead us to the rich theory of abelian categories.

Definition 6.5. A category $\mathscr{C}$ is a Ab-*category* if $\mathscr{C}$ is enriched in the category $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$.

This is equivalent to the statement that for all $A, B \in \mathscr{C}$ the homset $\mathscr{C}(A, B)$ is an abelian group and composition of maps is $\mathbb{Z}$-bilinear.

Definition 6.6. A category $\mathscr{C}$ is said to have *direct sums* if $\mathscr{C}$ has products and coproducts and for all finite index set I with $X_i \in \mathscr{C}$ for all $i \in I$ then

$$\coprod_{i \in I} X_i \to \prod_{i \in I} X_i$$

is an isomorphism.

Notation. We will denote the direct sum by $\oplus$ instead of the product/coproduct symbol.

Definition 6.7. An Ab-category $\mathscr{C}$ is an *additive category* if it has finite direct sums.

It is a well known fact that Vect_k direct sums a more surprising fact is that $\mathrm{Rep}\, G$ has.

Proposition 6.8. *If G is a group then* $\mathrm{Rep}\, G$ *has direct sums.*

Proof. Let $V, W \in \mathrm{Rep}\, G$. Define the representation $\rho_{V \oplus W} : G \to \mathrm{Aut}(V \oplus W)$ to be the composite

$$G \xrightarrow{\Delta} G \times G \xrightarrow{\rho_V \times \rho_W} \mathrm{Aut}\, V \times \mathrm{AutW} \xrightarrow{\times} \mathrm{Aut}(V \oplus W)$$

This clearly defines a representation. To see this is indeed a product in $\mathrm{Rep}\, G$ let X be a representation of G and $f : X \to V$ and $h : X \to W$ be G-linear maps. As these are in particular linear maps, they induce a unique linear map $t : X \to V \oplus W$ satisfying the universal property in $\mathrm{Vect}_{\mathbb{C}}$, this is in fact G-linear since if $g \in G$

$$\begin{aligned} t(\rho_X(g)(x)) &= (f(\rho_X(g)(x)), h(\rho_X(g)(v))) \\ &= (\rho_V(g)(f(x)), \rho_W(g)(h(x))) = \rho_{V \oplus W}(g)(t(x)). \end{aligned}$$

Showing that t is G-linear. The argument showing that this is also a coproduct is similar and therefor left out. ☐

Definition 6.9. Let $\mathscr{C}$ and $\mathscr{D}$ be Ab-categories. A functor $F \colon \mathscr{C} \to \mathscr{D}$ is *additive* if for all $X, Y \in \mathscr{C}$ the map

$$F \colon \mathscr{C}(X, Y) \to \mathscr{D}(F(X), F(Y))$$

is a group homomorphism.

Remark. It is clear that additive functors are exactly the Ab enriched functors.

Definition 6.10. Let $\mathscr{C}$ be a category an object $A \in \mathscr{C}$ is said to be *initial* if for all $B \in \mathscr{C}$ there exist precisely one map $A \to B$. Dually A is said to be *terminal* if there exist precisely one map $B \to A$. If A is both terminal and initial A is called a *zero* object.

It is easy to see that initial, terminal and zero objects are unique up to isomorphism. We therefor talk about the zero object 0.

Example 6.11. The category Vect_k has zero objects, namely the zero dimensional vector space 0. This is easily extended to $\mathrm{Rep}\, G$ for some group G.

Definition 6.12. Let $\mathscr{C}$ be a category with a zero object 0 and the data $f : A \to B$ in $\mathscr{C}$. Then the pair $K \in \mathscr{C}$ and $\ker f : K \to A$ is called the *kernel* of f if

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \ker f}\uparrow & & \uparrow \\ K & \longrightarrow & 0 \end{array}$$

commutes and it holds that for any other pair $(K', k : K' \to A)$ such that fk factors through 0 there exists a unique map $h : K' \to K$ such that

$$\begin{array}{ccccc} K & \xrightarrow{\ker f} & A & \xrightarrow{f} & B \\ & \nwarrow^{\exists! h} & {\scriptstyle k}\uparrow & & \uparrow \\ & & K' & \longrightarrow & 0 \end{array}$$

commutes. We will usually also denote K by $\ker f$.

We say that $\mathscr{C}$ has kernels is for all maps $f : A \to B$ in has a kernel.

In classical algebraic settings the inclusion from a kernel is typically injective. There is a similary result for categorical kernels namely

Proposition 6.13. *Let $\mathscr{C}$ be an additive category and $f\colon A \to B$ a map in $\mathscr{C}$. If f has a kernel then the map $i\colon \ker f \to A$ is a monomorphism.*

Proof. If for $C \overset{h}{\underset{g}{\rightrightarrows}} \ker f \xrightarrow{i} A$ $ih = ig$. Then the composite $fih = fig = 0$, hence by the universal property of the kernel there exists a unique map $z\colon C \to \ker f$ such that $iz = ih = ig$. By the uniqueness of z it holds that $z = h = g$. Hence i is a monomorphism. □

Definition 6.14. Let $\mathscr{C}$ be a category with a zero object 0 and $A, B \in \mathscr{C}$ with a map $f : A \to B$. The *cokernel* of f is an object C with a map $c : B \to C$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow{\scriptstyle c} \\ 0 & \longrightarrow & C \end{array}$$

commutes universally i.e. such that for any object X and map $h : B \to X$ that factors through 0 their exists a unique map $g : C \to X$ making

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & & \\ \downarrow & & \downarrow{\scriptstyle h} & \searrow^{c} & \\ 0 & \longrightarrow & X & \underset{g}{\longleftarrow} & C \end{array}$$

commute. The category $\mathscr{C}$ is said have cokernels if all maps has a cokernel.

In the section on semi-simple categories we will prove Schur's lemma, to do this we will need the following lemmata.

Lemma 6.15. *Let $\mathscr{C}$ be an additive category and $f\colon A \to B$ a map in $\mathscr{C}$.*

1. *f is monic if and only if for all $g\colon X \to A$ if $fg = 0$ then $g = 0$.*

2. *f is epic if and only if for all $h\colon B \to K$ if $hf = 0$ then $h = 0$.*

Proof. Let $\mathscr{C}$ be an additive category and $f\colon A \to B$ be a map in $\mathscr{C}$.

1. f being monic is equivalent to for all $X \in \mathscr{C}$ the map

$$\mathscr{C}(X, A) \to \mathscr{C}(X, B)$$
$$g \longmapsto fg$$

being injective. However $\mathscr{C}$ is additive hence the homs are abelian group and the induced map is a group homomorphism. Thus the map of homs induced by f is injective if and only if for all $g \in \mathscr{C}(X, A)$ it holds that if $fg = 0$ then $g = 0$.

2. In a similar fashion f is epic if and only map induced by precomposing is injective. We can therefor by the previous argument conclude that f is epic if and only for all $h \in \mathscr{C}(B, K)$ if $hf = 0$ then $f = 0$.

□

Lemma 6.16. *Let $\mathscr{C}$ be an additive category and $f : A \to B$ a map in $\mathscr{C}$.*

1. If f has a kernel then f is monic if and only if $\ker f = 0$.

2. If f has a cokernel then f is epic if and only if $\operatorname{coker} f = 0$.

Proof. Let $f : A \to B$ be a map in an additive category $\mathscr{C}$.

1. If $f : A \to B$ has a kernel and f is monic then for the inclusion $i : \ker f \to A$ it holds that $fi = f0 = 0$ thus since f is monic $i = 0$. If $i : \ker f \to A = 0$ then if for $g : X \to A$ it holds that $fg = 0$ by the universal property of the kernel their exists a map $h : X \to \ker f$ such that $g = ih = 0$, by assumption $i = 0$ thus $h = 0$.

2. The proof for the 2nd statement is analogous and therefore omitted.

□

We will not show that Vect_k is abelian but the next proposition will show the method one would use to go about proving this.

Proposition 6.17. *The category* Vect_k *has kernels. With* $\ker f = \{v \in V \mid f(v) = 0\}$ *and the inclusion map being the kernel of a linear map $f : V \to W$.*

Proof. Let $V, W \in \mathrm{Vect}_k$ and $f : V \to W$ be a linear map. Then $fi : \ker f \to B$ clearly factors through 0. Now let $(K, h : K \to A)$ be another pair such that fh factors through 0. Thus for $v \in K$ then since $fh(v) = 0$ we know that $h(v) \in \ker f$. We then define $g : K \to \ker f$ by $g(v) = h(v)$. This is clearly well defined and linear by the linearity of h. It is also the case that $ig = h$. Now to see that g is unique remember that the inclusion is injective, which is exactly the monos of Vect_k, showing the uniqueness of g. □

This proof tells us that the usual algebraic notion of a kernel is exactly the previously defined notion. While out of the scope of this project this proof generalises to the category $R\,\mathrm{Mod}$ of R-modules and R-linear maps. Before diving further in to the definition of an abelian category, we will take a stint into representation theory, if only to define the appropriate notions to show that $\mathrm{Rep}\,G$ is abelian.

Definition 6.18. Let G be a group and (V, ρ) a representation of G. A subspace W of V is called an *invariant subspace* of V if for all $v \in W$ and $g \in G$ we have $\rho(g)(v) \in W$. An invariant subspace is canonically a representation with the representation given by $\rho_W : G \to \mathrm{Aut}\,W$ where $\rho_W(g) = \rho(g)\mid_W$. Such a representation is called a *subrepresentation* of (V, ρ).

A not so surprising result is that the zero dimensional vector space equiped with the trivial representation is the zero object of $\mathrm{Rep}\,G$.

Proposition 6.19. *Let $V, W \in \operatorname{Rep} G$ for some group G. If $f : V \to W$ is G-linear then* $\ker f$ *and* $\operatorname{im} f$ *are invariant subspaces of V and W.*

We will only show the proof for the kernel. The argument for the image is similar.

Proof. Let $v \in \ker f$ then for $g \in G$

$$f(\rho_V(g)(v)) = \rho_W(g)(f(v)) = \rho_W(g)(0) = 0.$$

Thus $\rho_V(g)(v) \in \ker f$. □

It follows from construction that the inclusion of a subrepresentation is G-linear. This lets us conclude the following.

Proposition 6.20. *If G is a group then* $\operatorname{Rep} G$ *has kernels.*

Proof. Since $\mathrm{Vect}_{\mathbb{C}}$ has kernels and the inclusion is G-linear then $(\ker f, i : \ker f \to V)$ is the kernel of any G-linear map $f : V \to W$. □

Additionally we are able to define a notion of quotient representation.

Definition 6.21. Let V be a representation of a group G and W an invariant subspace. We define the *quotient representation* to be the pair $(V/W, \rho_{V/W})$ where $\rho_{V/W}(g)(v+W) = \rho(g)(v)+W$.

To see this is well defined let v, v' be elements of some coset $v + W$. Then $v - v' \in W$ thus $\rho(g)(v - v') \in W$. There for $\rho(g)(v) + W = \rho(g)(v') + W$. Just like with the inclusion it is evident that the canonical projection $p : V \to V/W$ is G-linear. This will be important in showing that $\operatorname{Rep} G$ has cokernels. To realize that $\operatorname{Rep} G$ has cokernels, for a G-linear map of representations $f : V \to W$ inspect the quotient $W/\operatorname{Im} f$ with the projection $p : W \to W/\operatorname{Im} f$.

6.3 Abelian categories

We will now introduce the theory of abelian categories and additionally we will show that the category $\operatorname{Rep} G$ is an abelian category.

Definition 6.22. A category $\mathscr{C}$ is *abelian* if

- $\mathscr{C}$ is Additive.
- $\mathscr{C}$ has kernels and cokernels.
- Every mono is the kernel of its cokernel and every epi is the cokernel of its kernel.

Theorem 6.23. *If G is a group then* $\operatorname{Rep} G$ *is abelian.*

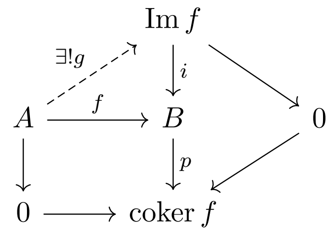
Proof. To see that $\operatorname{Rep} G$ is Ab-enriched one need only realise that composition of G-linear maps is G-linear thus the map induced by the tensor product of vector spaces is G-linear, therefor since $\mathrm{Vect}_{\mathbb{C}}$ is enriched in $\mathrm{Vect}_{\mathbb{C}}$, in particular Ab, then so is $\operatorname{Rep} G$. By Proposition 4.5 $\operatorname{Rep} G$ has direct sums thus $\operatorname{Rep} G$ is additive and by proposition 4.12 it has kernels and a similar argument shows that it has cokernels. At last since every G-linear map is in particular a linear map, then since $\mathrm{Vect}_{\mathbb{C}}$ satisfies the third condition it follows that $\operatorname{Rep} G$ also is. Showing that $\operatorname{Rep} G$ is abelian. □

We have now shown that $\operatorname{Rep} G$ is an abelian monoidal category.

Definition 6.24. Let $\mathscr{C}$ be an abelian category. For $f \colon A \to B$ we define $\operatorname{Im} f := \ker(\operatorname{coker} f)$.

Proposition 6.25. *Any map $f \colon A \to B$ in an abelian category $\mathscr{C}$ factors through* $\operatorname{Im} f$.

Proof. Consider the diagram:



This diagram commutes by the definition of cokernels and since $\operatorname{Im} f$ is a kernel. In particular since $pf = 0$ there exists a unique map $g\colon A \to \operatorname{Im} f$ such that $ig = f$. □

We can now define exactness as follows

Definition 6.26. Let $\mathscr{C}$ be an abelian category and the sequence of maps

$$\dots \longrightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \dots$$

is said to be *exact at degree n* if $\operatorname{Im} f_{n-1} = \ker f_n$. It is called *exact* if it is exact at every degree. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

Definition 6.27. Let $\mathscr{C}, \mathscr{D}$ be abelian categories, $F\colon \mathscr{C} \to \mathscr{D}$ an additive functor and

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence.

- The functor F is left exact if $0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$ is exact.
- The functor F is right exact if $F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$ is exact.
- The functor F is exact if its both left and right exact.

7 Simples and semi-simples

Given an abelian category $\mathscr{C}$ a question one might ask is if all objects can be written as the direct sum of more well understood objects. This property is called semi-simplicity and will be the subject of this section.

Definition 7.1. Let $\mathscr{C}$ be an abelian category and $A \in \mathscr{C}$. An object $B \in \mathscr{C}$ is called a subobject of A if there exists a monomorphism $B \to A$.

It is clear that the subobjects in Vect_k are subspaces and this does in fact extend to subrepresentations in $\operatorname{Rep} G$.

Proposition 7.2. *Let G be a group if W is a subrepresentation of V then W is a subobject of V*

Proof. The inclusion map is a G-linear monomorphism. □

Definition 7.3. Let $\mathscr{C}$ be an abelian category and $A \in \mathscr{C}$.

- A is *simple* if the only subobjects of A are 0 and A itself.
- A is *semi-simple* if $A \cong \bigoplus_{i \in I} S_i$ where S_i is simple for all $i \in I$.

- The category $\mathscr{C}$ is called *semi-simple* if all objects in $\mathscr{C}$ are semi-simple.

Remark. It is a well known fact that the category $\mathrm{FinVect}_k$ of finite dimensional vector spaces and linear maps is semi simple.

The rest of this section will be dedicated to showing that if G is a finite group then $\operatorname{Rep} G$ is semi-simple. This is however not as easy an fact to show as in the case of $\mathrm{FinVect}_k$. To do this we will use the well known fact that any finite dimensional vector space V can be equipped with an inner product.
The following definition and results on representation theory is inspired by [Tel05].

Definition 7.4. Let V be a representation of a group G equipped with an inner product $\langle -, - \rangle$. We say that $\langle -, - \rangle$ is unitary if for all $g \in G$

$$\langle -, - \rangle = \langle \rho(g)(-), \rho(g)(-) \rangle$$

While not every inner product is unitary we can always construct an unitary inner product.

Theorem 7.5 (Weyl's unitary trick)**.** *If V is a representation of a finite group G equiped with an inner product $\langle -, - \rangle$ then*

$$\langle -, - \rangle' = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)(-), \rho(g)(-) \rangle$$

is an unitary inner product on the representation V.

Proof. It is clear that linearity in the first argument and conjugate linearity in the second argument is preserved by this construction. Therefor assume for $v \in V$ that $\langle v, v \rangle' = 0$ then

$$\frac{1}{|G|} \sum_{g \in G} \langle \rho(g)(v), \rho(g)(v) \rangle = 0 \Leftrightarrow \sum_{g \in G} \langle \rho(g)(v), \rho(g)(v) \rangle = 0 \Leftrightarrow \langle \rho(g)(v), \rho(g)(v) \rangle = 0 \quad \forall g \in G,$$

showing that $\langle -, - \rangle'$ is positive definite, since $\rho(g)$ is an isomorphism for all $g \in G$ and the since addition and multiplication of positive positive numbers preserve the sign. To see that $\langle -, - \rangle'$ is unitary let $h \in G$ then

$$\begin{aligned} \langle \rho(h)(-), \rho(h)(-) \rangle' &= \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)(\rho(h)(-)), \rho(g)(\rho(h)(-)) \rangle \\ &= \frac{1}{|G|} \sum_{g \in G} \langle \rho(gh)(-), \rho(gh)(-) \rangle \\ &\stackrel{*}{=} \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)(-), \rho(g)(-) \rangle \end{aligned}$$

The equality at "*" follows from the fact that the action of left multiplication is free invariant and transitive on G. □

The proof that $\operatorname{Rep} G$ is semi-simple if G is finite now comes in two pieces. Showing that every for invariant subspace the orthogonal complement of that subspace is also invariant and using this to show that every representation of a finite group is semi-simple.

Theorem 7.6. *Let V be a representation of a finite group G with unitary inner product $\langle -, - \rangle$ and W an invariant subspace of V then the orthogonal complement $W^\perp$ is an invariant subspace.*

Proof. Let $v \in W^\perp$ then $\langle v, v' \rangle = 0 = \langle \rho(g)(v), \rho(g)(v') \rangle$ for all $g \in G$ and all $v' \in W$. Then since $\rho(g^{-1})(w) \in W$ for all $w \in W$ let $v' = \rho(g^{-1})(w)$. By this we can conclude that

$$\langle \rho(g)(v), \rho(g)(v') \rangle = \langle \rho(g)(v), w \rangle = 0$$

for all $w \in W$. Hence $W^\perp$ is also an invariant subspace. □

Corrolary. *If V is a finite dimensional representation of a finite group G then V is semi-simple in* $\operatorname{Rep} G$.

Proof. We proceed by induction on the dimension n of V. If $n = 1$ then V is simple. Assume $\dim V = n$. We may assume that V is an unitary representation by Weyl's trick. We may also assume that V is not simple hence V has an invariant subspace W then by theorem 6.6 that W's orthogonal complement $W^\perp$ is also an invariant subspace and $\dim W < n$ and $\dim W^\perp < n$ thus by the induction assumption $W = \bigoplus_{i\in I} S_i$ and $W^\perp = \bigoplus_{j\in J} S_j$ with S_i and S_j simple for all $i \in I$ and $j \in J$. Therefore

$$V = W \oplus W^\perp = \bigoplus_{i\in(I \coprod J)} S_i.$$

Hence we conclude that V is semi-simple. □

Finally we will show Schur's lemma in terms of simple objects in an abelian therefore we can conclude this section with the fact that $\operatorname{Rep} G$ is semi-simple.

Theorem 7.7 (Schur's lemma)**.** *Let $\mathscr{C}$ be an abelian category and let $A, B \in \mathscr{C}$ be simple objects. If $f \colon A \to B$ then f is an isomorphism or $f = 0$.*

Proof. Let $f \colon A \to B$ be a non-zero map. Consider the diagram:

$$\ker f \xrightarrow{k} A \xrightarrow{p} \operatorname{Im} f \xrightarrow{i} B \xrightarrow{c} \operatorname{coker} f, \qquad f = i \circ p \colon A \to B$$

Since k is a monomorphism $\ker f$ is a subobject of A but f is non-zero so $k = 0$ thus f is monic. Then $\operatorname{Im} f = A$ and $f = i$ but $\operatorname{Im} f$ is a kernel thus f is monic. Thus since B is simple $A \cong 0$ or $A \cong B$ however since f is non-zero $A \cong B$. □

In fact this gives a good characterization of homs on simple objects.

Corrolary. *If $A \in \mathscr{C}$ is simple then $\mathscr{C}(A, A)$ is a division ring.*

This fact is clear and the proof is omitted. We will later rephrase this corollary in a specific case.

8 Linear and tensor categories

In this chapter we will define linear and tensor categories, these notions will tie together the teory of monoidal categories and abelian categories.

Definition 8.1. A category $\mathscr{C}$ is linear if $\mathscr{C}$ is an abelian category such that the homspaces are complex vector spaces.

We have shown that for a group G the category $\operatorname{Rep} G$ is an abelian category. It is however in fact linear.

Theorem 8.2. *If G is a group then the category* $\operatorname{Rep} G$ *is linear.*

Proof. In the proof showing $\operatorname{Rep} G$ is abelian we used that vector spaces are in particular abelian groups. □

This additional structure on the homs in $\operatorname{Rep} G$ actually expands to Schur's lemma to the following case:

Proposition 8.3 (Schur's Lemma in a linear category.)**.** *If A is a simple object in a linear category $\mathscr{C}$ then* $\mathrm{End}(A)$ *is a division algebra.*

The proof is essentially the same as in the case of abelian categories and therefor excluded once again. In the case where k is an algebraically closed field it is a well known fact that any division algebra A over k must be isomorphic to k. See for example [Coh12] for a proof.

Definition 8.4. Let $\mathscr{C}, \mathscr{D}$ be linear categories an additive functor is linear if the map

$$F\colon \mathscr{C}(A,B) \to \mathscr{D}(F(A), F(B))$$

is linear for all $A, B \in \mathscr{C}$

Now to connect the notions of monoidality and linearity we have the following definition:

Definition 8.5. a linear monoidal category $(\mathscr{C}, \otimes, 1)$ is a tensor category if the bifunctor $\otimes$ is linear on hom sets and 1 is simple.

Definition 8.6. A functor $F\colon \mathscr{C} \to \mathscr{D}$ between tensor categories is *tensor* if it is linear and monoidal.

Definition 8.7. A functor $F\colon \mathscr{C} \to \mathrm{Vect}_{\mathbb{C}}$ between tensor categories $\mathscr{C}, \mathscr{D}$ is a *fiber functor* if it is tensor and exact

Example 8.8. The forgetful functor is fiber functor.

9 Fusion categories

In this chapter we take the final steps towards defining symmetric fusion categories and finish showing that $\mathrm{Rep}\, G$ is symmetric fusion.

Definition 9.1. Let $(\mathscr{C}, \otimes, 1)$ be a tensor category. The category $\mathscr{C}$ is *fusion* if $\mathscr{C}$ is semi-simple, rigid, the unit 1 is a simple object and have finitely many isomorphism classes of simple objects

Remark. We say that a fusion category $\mathscr{C}$ is braided/symmetric if the monoidal structure on $\mathscr{C}$ is braided/symmetric.

We have shown that during this project shown that for a finite group G the category $\mathrm{Rep}\, G$ is a symmetric fusion category.

10 Yoneda lemma

In this section we will prove the Yoneda lemma, a classic theorem of category theory. Both in the standard case and in the case of an enriched a category. The Yoneda lemma captures a great deal of categorical philosophy, namely that objects are determined uniquely by their relations to other objects.

10.1 The classical Yoneda lemma

Notation. For $\mathscr{C}, \mathscr{D}$ categories, We denote functor category from $\mathscr{C}$ to $\mathscr{D}$ by $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ and for $F, G \in \mathrm{Fun}(\mathscr{C}, \mathscr{D})$ we denote the hom-set from F to G by $\mathrm{Nat}(F, G)$.

Theorem 10.1 (Yoneda Lemma). *Let $\mathscr{C}$ be a category and $F\colon \mathscr{C} \to \mathrm{Set}$ be a functor and $X \in \mathscr{C}$ an object. Then their is a bijection*

$$y\colon \mathrm{Nat}(\mathscr{C}(X,-),F) \xrightarrow{\cong} F(X)$$

given by $\alpha\colon \mathscr{C}(X,-) \Rightarrow F \mapsto \alpha_X(\mathrm{id}_X)$. Additionally this bijection is natural in both X and F.

Proof. We start by constructing an inverse $\tilde{y}$ to y. For $Z \in \mathscr{C}$ and $f \in \mathscr{C}(X,Z)$ we get from the naturality of α that

$$\begin{array}{ccc} \mathscr{C}(X,X) & \xrightarrow{\alpha_X} & F(X) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \mathscr{C}(X,Y) & \xrightarrow{\alpha_Z} & F(Z) \end{array}$$

commutes. In particular $F(f)(\alpha_X(\mathrm{id}_X)) = \alpha_Z(f)$. We then define $\tilde{y}\colon F(X) \to \mathrm{Nat}(\mathscr{C}(X,-),F)$ where we map $s \in F(X)$ to the natural transformation with components

$$\begin{aligned} \beta_Z\colon \mathscr{C}(X,Z) &\longrightarrow F(Z) \\ g &\longmapsto F(g)(s) \end{aligned}$$

Then we check that these are mutually inverse $(\tilde{y}y)(\alpha) = \tilde{y}(\alpha_X(\mathrm{id}_X))$. On components this is given by $\beta_Z(f) = F(f)(\alpha_X(\mathrm{id}_X)) = \alpha_Z(f)$ and thus we conclude $(\tilde{y}y)(\alpha) = \alpha$. We also see that

$$(y\tilde{y})(s) = F(\mathrm{id}_X)(s) = \mathrm{id}_{F(X)}(s) = s.$$

We now prove that this is natural in F and X. We define the following two functors

$$E, N\colon \mathscr{C} \times \mathrm{Cat}(\mathscr{C},\mathrm{Set}) \to \mathrm{Set}$$

Defined on objects $X \in \mathscr{C}$ and $F \in \mathrm{Cat}(\mathscr{C},\mathrm{Set})$ as

$$N(X,F) = \mathrm{Nat}(\mathscr{C}(X,-),F), E(X,F) = F(X)$$

and on morphisms $(f,\alpha)\colon (X,F) \to (Y,G)$ for $f \in \mathscr{C}(X,Y)$ and $\mathrm{Nat}(F,G)$ is given by the composites

$$\begin{aligned} N(f,\alpha)(\beta)_Z &= \mathscr{C}(Y,Z) \xrightarrow{f_*} \mathscr{C}(X,Z) \xrightarrow{\beta_Z} F(Z) \xrightarrow{\alpha_Z} G(Z) \\ E(f,\alpha) &= G(f)\alpha_X = \alpha_Y F(f) \end{aligned}$$

We now check that

$$\begin{array}{ccc} \mathrm{Nat}(\mathscr{C}(X,-),F) & \xrightarrow{y_{X,F}} & F(X) \\ {\scriptstyle N(f,\alpha)}\downarrow & & \downarrow{\scriptstyle E(f,\alpha)} \\ \mathrm{Nat}(\mathscr{C}(Y,-),G) & \xrightarrow{y_{Y,G}} & G(Y) \end{array}$$

commutes. Let $\beta \in \mathrm{Nat}(\mathscr{C}(X,-),F)$, the first composite is

$$\beta \overset{y_{X,F}}{\longmapsto} \beta_X(\mathrm{id}_X) \overset{E(f,\alpha)}{\longmapsto} (\alpha_Y F(f))(\beta_X(\mathrm{id}_X)) = \alpha_Y(\beta_Y(f)) = (\alpha_Y\beta_Y)(f)$$

and the second composite gives us

$$\beta \overset{N(f,\alpha)}{\longmapsto} N(f,\alpha)(\beta) \overset{y_{Y,G}}{\longmapsto} N(f,\alpha)(\beta)(\mathrm{id}_Y) = (\alpha_Y\beta_Y)(f_*(\mathrm{id}_Y)) = \alpha_Y\beta_Y(f)$$

hence y is natural in F and X. □

Before we prove a next corollary we will prove the following lemma.

Lemma 10.2. *If $F\colon \mathscr{C} \to \mathscr{D}$ is a fully faithfull functor then F reflects isomorphisms.*

Proof. Let $X, Y \in \mathscr{C}$ with $F(X) \cong F(Y)$. Suppose $f\colon F(X) \to F(Y)$ is this isomorphism. Then by the fullness of F their exists a map $g\colon X \to Y$ such that $F(g) = f$. Additionally let $g'\colon Y \to X$ be the map such that $F(g') = f^{-1}$. Then by functorality we get

$$F(g'g) = F(g')F(g) = f^{-1}f = \mathrm{id}_{F(X)} = F(\mathrm{id}_X)$$

thus by the faithfulness of F $g'g = \mathrm{id}_X$. One similarly sees that $gg' = \mathrm{id}_Y$. □

Corrolary. *Let $\mathscr{C}$ be a category and $X, Y \in \mathscr{C}$ then $X \cong Y$ if and only if $\mathscr{C}(X, -) \cong \mathscr{C}(Y, -)$*

Proof. The if part is clear. For the only if the Yoneda lemma implies that the Yoneda embedding

$$\begin{aligned} \mathscr{Y}: \mathscr{C} &\to \mathrm{Fun}(\mathscr{C}^{op}, \mathrm{Set}) \\ \mathscr{C} \ni X &\mapsto \mathscr{C}(X, -) \\ f &\mapsto f_* \end{aligned}$$

is fully faithfull thus it reflects isomorphisms. Hence $X \cong Y$. □

This concludes are view on the Yoneda lemma. This will act as the recipe for which we prove the $\mathrm{Vect}_{\mathbb{C}}$ enriched Yoneda lemma.

10.2 The $\mathrm{Vect}_{\mathbb{C}}$-enriched Yoneda lemma

Unfortunately we the functors of particular interest for us are enriched functors. There is however a version of the Yoneda lemma compatible with the theory of enriched categories. This subsection will provide a proof of the case in which we enrich over $\mathrm{Vect}_{\mathbb{C}}$.

Proposition 10.3. *If $F, G\colon \mathscr{C} \to \mathrm{Vect}_{\mathbb{C}}$ are functors then $\mathrm{Nat}(F, G)$ is a vector space.*

We will only sketch the proof this, since checking all the other axioms is essentially the same.

Proof. Let $\alpha, \beta \in \mathrm{Nat}(F, G)$ and for all $X \in \mathrm{Vect}_{\mathbb{C}}$ define $(\alpha + \beta)_X :- \alpha_X + \beta_X$. If $f \in \mathscr{C}(X, Y)$ then

$$(\alpha_Y + \beta_Y)(F(f)) = \alpha_Y(F(f)) + \beta_Y F(f) = G(f)\alpha_X + G(f)\alpha_X = G(f)(\alpha_X + \beta_X)$$

Thus showing the sum of two natural transformations again is a natural transformation. □

If we consider a $\mathrm{Vect}_{\mathbb{C}}$-enriched category $\mathscr{C}$ and an object $X \in \mathscr{C}$, then the functors $\mathscr{C}(X, -)$ and $\mathscr{C}(-, X)$ are $\mathrm{Vect}_{\mathbb{C}}$-enriched functors. In particular there is a version of the Yoneda lemma that applies to the hom functors of $\mathrm{Vect}_{\mathbb{C}}$ enriched categories.

Theorem 10.4 (The $\mathrm{Vect}_{\mathbb{C}}$-enriched Yoneda lemma)**.** *Let $\mathscr{C}$ be a $\mathrm{Vect}_{\mathbb{C}}$ category, $F\colon \mathscr{C} \to \mathrm{Vect}_{\mathbb{C}}$ a $\mathrm{Vect}_{\mathbb{C}}$-functor and $X \in \mathscr{C}$. Then there is an isomorphism of vector spaces*

$$\mathrm{Nat}(\mathscr{C}(X, -), F) \cong F(X)$$

which sends natural transformations α to $\alpha_X(\mathrm{id}_X)$.

The proof is essentially the same as for the non enriched Yoneda lemma, the only addition is to check that the maps are in the proof are indeed linear. This is easily done. Therefor we omit the proof. This makes the enriched Yoneda embedding fully faithful, hence it reflects isomorphisms.

11 Tensored categories

in this section we will present the theory of tensored categories. We will state the theorems of this section in the context of a general a closed symmetric monoidal category $(\mathscr{C}, \otimes, 1)$ and a $\mathscr{C}$-category $\mathscr{D}$. However ever since we have not proven the general enriched Yoneda lemma, the reader will have to take on faith that these can be proven in this context. We will only need the case in which $\mathscr{C} = \mathrm{Vect}_k$, which was proven earlier.

Definition 11.1. Let $(\mathscr{C}, \otimes, 1)$ be closed symmetric monoidal category and $\mathscr{D}$ be a $\mathscr{C}$-category. The category $\mathscr{D}$ is tensored if for all $a \in \mathscr{C}$ and m their exists a object $a \cdot m \in \mathscr{D}$ such that for all $n \in \mathscr{D}$ their is a isomorphism

$$\mathscr{D}(a \cdot m, n) \cong \mathscr{C}(a, \mathscr{D}(m, n)).$$

We can now consider the functor

$$\begin{aligned} M \colon \mathscr{C} &\to \mathrm{End}\,\mathscr{D} \\ a &\mapsto a \cdot - \end{aligned}$$

It turns that M is a monoidal functor, which provides us with a number of interesting lemmata.

Theorem 11.2. *Let $\mathscr{D}$ be a tensored $\mathscr{C}$-category. The functor $M \colon \mathscr{C} \to \mathrm{End}\,\mathscr{D}$ is monoidal.*

Proof. Since $\mathrm{End}\,\mathscr{D}$ is a strict monoidal category, we will only show the existens of isomorphisms

$$\begin{aligned} \left(c' \otimes c\right) \cdot m &\xrightarrow{\cong} c' \cdot (c \cdot m) \\ m &\xrightarrow{\cong} 1 \cdot m \end{aligned}$$

for all $c', c \in \mathscr{C}$ and $m \in \mathscr{D}$. Consider the isomorphisms on homs

$$\begin{aligned} \mathscr{D}\left((c' \otimes c) \cdot m, k\right) &\cong \mathscr{C}(c' \otimes c, \mathscr{D}(m, k)) \\ &\cong \mathscr{C}(c', \mathscr{C}(c, \mathscr{D}(m, k))) \\ &\cong \mathscr{C}(c', \mathscr{D}(c \cdot m, k)) \\ &\cong \mathscr{D}(c' \cdot (c \cdot m), k). \end{aligned}$$

These isomorphisms follow from repeated use of the tensoring identity and the hom-tensor adjunction in $\mathscr{C}$. It now follows from the yoneda lemma that

$$(c' \otimes c) \cdot m \cong c' \cdot (c \cdot m)$$

naturally in c', c and m. Similary it follows

$$\mathscr{D}(1 \cdot m, n) \cong \mathscr{C}(1, \mathscr{D}(m, n)) \cong \mathscr{D}(m, n)$$

hence by the yoneda lemma it follows that

$$1 \cdot m \cong m$$

naturally in m. □

We have now established that the functor M is a monoidal functor, if equipped with the isomorpisms of theorem 12.2. We can now apply the theory of monoidal categories and functors we have developed in earlier sections of the project.

Lemma 11.3. *If $\mathscr{D}$ is a category tensored in $\mathscr{C}$ and $c \in \mathscr{C}$ has a right dual c^* then then there is an isomorphism*

$$\mathscr{D}(c \cdot m, n) \cong \mathscr{D}(m, c^* \cdot n)$$

natural in $m, n \in \mathscr{D}$. More consisely we have an adjunction

$$c \cdot - \dashv c^* \cdot -.$$

Proof. The functor $M \colon \mathscr{C} \to \operatorname{End} \mathscr{D}$ is monoidal thus $M(c)$ has a right dual $M(c^*)$. Hence $c^* \cdot -$ is a right dual of $a \cdot -$. Then since the right duals in $\operatorname{End} \mathscr{D}$ are particularly the right adjoints. Hence we get the proposed natural isomorphism. □

Using this lemma we can now prove the following.

Lemma 11.4. *If $\mathscr{D}$ is a category tensored in $\mathscr{C}$, and $c \in \mathscr{C}$ has a dual c^* we have an isomorphism*

$$\mathscr{D}(m, c \cdot n) \cong c \otimes \mathscr{D}(m, n)$$

natural in m, n and c.

Proof. Consider the isomorphisms

$$\begin{aligned}
\mathscr{C}(x, \mathscr{D}(m, c \cdot n)) &\cong \mathscr{C}(x, \mathscr{D}(c^* \cdot m, n)) \\
&\cong \mathscr{C}(x, \mathscr{C}(c^*, \mathscr{D}(m, n))) \\
&\cong \mathscr{C}(x, \mathscr{C}(1, c \otimes \mathscr{D}(m, n))) \\
&\cong \mathscr{C}(x, c \otimes \mathscr{D}(m, n)).
\end{aligned}$$

These all follow from the various results proved in this chapter. □

In particular a symmetric fusion category is tensored over $\mathrm{Vect}_{\mathbb{C}}$.

Theorem 11.5. *If $\mathscr{A}$ is a symmetric fusion category then A is tensored over $\mathrm{Vect}_{\mathbb{C}}$ with $V \cdot X = X^{\oplus \dim V}$.*

Proof. Consider the isomorphisms

$$\begin{aligned}
\mathscr{A}(X^{\oplus \dim V}, Y) &\cong \bigoplus_{\dim V} \mathscr{A}(X, Y) \\
&\cong \bigoplus_{\dim V} \mathrm{Vect}_{\mathbb{C}}(\mathbb{C}, \mathscr{A}(X, Y)) \\
&\cong \mathrm{Vect}_{\mathbb{C}}(\mathbb{C}^{\oplus \dim V}, \mathscr{A}(X, Y)) \\
&\cong \mathrm{Vect}_{\mathbb{C}}(V, \mathscr{A}(X, Y))
\end{aligned}$$

The first isomorphisms follows from the fact that in an additive category, direct sums commute with the hom functors. The second follow by the definition of homs in a $\mathrm{Vect}_{\mathbb{C}}$ enriched category. The third is again the fact that homs and direct sums commute. The last is clear. □

This finishes the section on tensored categories.

12 Tannaka duality

In this section we will give the statement of the Tannaka duality and prove the tannaka duality for a special class of symmetric fusion categories. Additionally we will state and prove two reconstruction theorems one of which is essential to the version of Tannaka duality we prove in this project.

Theorem 12.1 (Tannaka duality theorem for symmetric fusion categories)**.** *If $\mathscr{A}$ is a symmetric fusion category and $F\colon \mathscr{A} \to \mathrm{Vect}_{\mathbb{C}}$ is a fiber functor. Then their is a monoidal equivalence of categories*

$$\Phi\colon \mathscr{A} \xrightarrow{\simeq} \mathrm{Rep}(\mathrm{Aut}^{\otimes}(F)).$$

The proof of this theorem was given by Deligne in [Del90].

12.1 The Tannaka reconstruction theorems

For a category $\mathscr{C}$ one can define $\mathscr{C}$-representations of a group G.

Definition 12.2. Let G be a group and consider the corresponding deelooping category $\underline{G}$ and let $\mathscr{C}$ be a category. A $\mathscr{C}$-representation of G is a functor $F\colon \underline{G} \to \mathscr{C}$.

It is easy to see that if $\mathscr{C} = \mathrm{Vect}_{\mathbb{C}}$ this corresponds to representations of G. Additionally its easy to see that a natural transformation from of two representations of G is exactly a G-linear map.

Definition 12.3. Let a group G and $\mathscr{C}$ a category. A G-equivariant map[2] between representations F, G is a natural transformation $\alpha\colon F \Rightarrow G$. Additionally the category $\mathrm{Rep}_{\mathscr{C}}\, G$ of $\mathscr{C}$-representations of a group G, we define to be the category:

$$\mathrm{Rep}_{\mathscr{C}}(G) \coloneqq \mathrm{Fun}(\underline{G}, \mathscr{C}).$$

This allows us to state the simplest version of Tannaka duality namely

Theorem 12.4 (Tannaka reconstruction theorem for Set-representations.)**.** *Let G be a group and*

$$U\colon\ \mathrm{Rep}_{\mathrm{Set}}(G) \to \mathrm{Set}$$

be the functor which sends $F\colon G \to \mathrm{Set}$ to $F(*)$ and acts trivially one morphisms. Then there is a group isomorphism

$$\mathrm{Aut}(U) = \mathrm{End}(U) \cong G.$$

Proof. Consider the Yoneda embedding

$$\mathscr{Y}: \underline{G} \to \mathrm{Rep}_{\mathrm{Set}}\, G$$

with $Y(*) = \underline{G}(*, -)$ and $Y(g)$ being the action of right multiplication by g. By the Yoneda lemma their exists a family of isomorphisms parametrized by $\rho \in \mathrm{Rep}_{\mathrm{Set}}(G)$

$$\tau_\rho\colon\ \mathrm{Nat}(\underline{G}(*, -), \rho) \cong \rho(*) = U(\rho)$$

which is natural in ρ, this also follows from the Yoneda lemma. Now by multiple applications of the Yoneda lemma it follows that

$$\begin{aligned} \mathrm{End}\, U &\cong \mathrm{Nat}(\mathrm{Nat}(\underline{G}(*, -)), \mathrm{Nat}(\underline{G}(*, -))) \\ &\cong \mathrm{Nat}^{op}(\underline{G}(*, -), \underline{G}(*, -)) \\ &\cong \underline{G}(*, *) = G \end{aligned}$$

Since G is a group it follows that $\mathrm{End}\, U = \mathrm{Aut}\, U$. □

[2] Typically we will just say G-map.

This proof only use that G is a group at end. Besides the conclusion this would hold for any monoid. In fact this is a special case of a more general reconstruction theorem. We are however only concerned with another special case of this theorem.

Definition 12.5. Let $\left(\mathscr{C}, \underset{\mathscr{C}}{\otimes}, 1\right)$ be a $\mathrm{Vect}_{\mathbb{C}}$-enriched monoidal category and (A, μ, e) be a monoid in $\mathscr{C}$. The *delooping category* of A is the $\mathrm{Vect}_{\mathbb{C}}$-enriched category $\underline{A}$ with $\mathrm{ob}(A) = \{*\}$ and $\underline{A}(*, *) = \mathscr{C}(A, A)$ and composition

$$\mu\colon A \underset{\mathscr{C}}{\otimes} A \to \mathscr{C}(A, A)$$

and identity

$$e\colon \mathbb{C} \to \mathscr{C}(A, A)$$

With this definition comes a generalized definition of a representation of a monoid object.

Definition 12.6. Let $\left(\mathscr{C}, \underset{\mathscr{C}}{\otimes}, 1\right)$ be a monoidal category and (A, μ, e) be a monoid in $\mathscr{C}$. We define the category of representations of A to be the category

$$\mathrm{Rep}\, A \coloneqq {}_{\mathrm{Vect}_{\mathbb{C}}} \mathrm{Fun}(\underline{A}, \mathrm{Vect}_{\mathbb{C}})$$

of $\mathrm{Vect}_{\mathbb{C}}$-enriched functors from $\underline{A}$ to $\mathrm{Vect}_{\mathbb{C}}$.

Proposition 12.7. *Let G be a group. Then there is an equivalence of categories*

$$\mathrm{Rep}\, G \simeq \mathrm{Rep}\, \mathbb{C}[G]$$

Proof. This is immediately clear from Theorem 1.14. □

We can now prove the Tannaka reconstruction theorem for $\mathbb{C}$-algebras.

Theorem 12.8 (Tannaka reconstruction theorem for $\mathbb{C}$-algebras)**.** *Let G be a finite group and*

$$U\colon\ \mathrm{Rep}\, \mathbb{C}[G] \to \mathrm{Vect}_{\mathbb{C}}$$

be the functor with $U(\rho) = \rho(*)$ and which acts trivially on morphisms. Then there is an isomorphism of vector spaces

$$\mathbb{C}[G] \cong \mathrm{End}\, U.$$

In particular this induces an equivalence of categories $\mathrm{Rep}\, \mathbb{C}[G] \simeq \mathrm{Rep}(\mathrm{End}\, U)$

This proof is analogous to the Set case and therefor we will skip some of the details.

Proof. Consider the Yoneda embedding

$$\mathscr{Y}\colon \underline{\mathbb{C}[G]} \to \mathrm{Rep}\, \mathbb{C}[G]$$

with $\mathscr{Y}(*) = \underline{\mathbb{C}[G]}(*, -)$ which acts on morphisms by right multiplication. From the enriched Yoneda lemma we get a natural isomorphism with components

$$\tau_\rho\colon\ \mathrm{Nat}(\underline{\mathbb{C}[G]}(*, -), \rho) \cong \rho(*) = U(\rho).$$

Thus by repeated use of the Yoneda lemma we conclude that

$$\begin{aligned} \mathrm{End}\, U &\cong \mathrm{Nat}(\mathrm{Nat}(\underline{\mathbb{C}[G]}(*, -), -), \mathrm{Nat}(\underline{\mathbb{C}[G]}(*, -), -)) \\ &\cong \mathrm{Nat}^{op}(\underline{\mathbb{C}[G]}(*, -), \underline{\mathbb{C}[G]}(*, -)) \\ &\cong \underline{\mathbb{C}[G]}(*, *) = \mathbb{C}[G] \end{aligned}$$

It is clear that isomorphic representing object gives rise to an equivalence of categories. □

This theorem essentially half of the ingredients to the Tannaka duality presented in the next subsection.

12.2 Tannaka duality for representations of finite groups

This section will be the conclusion of the project and will be dedicated to proving Tannaka duality for a special class of symmetric fusion categories namely the category of representations of finite groups.

Theorem 12.9 (Tannaka duality for representations of finite groups)**.** *Let G be a finite group and*

$$U\colon \operatorname{Rep}\mathbb{C}[G] \to \mathrm{Vect}_{\mathbb{C}}$$

With $U(\rho) = \rho(*)$ and which acts trivially on arrows. Then there exists an equivalence of categories

$$\operatorname{Rep} G \simeq \operatorname{Rep}(\operatorname{Aut}^{\otimes} U)$$

Proof. From Proposition 12.7 we get an equivalence

$$\operatorname{Rep} G \simeq \operatorname{Rep}\mathbb{C}[G].$$

Then from Theorem 12.8 we conclude that

$$\operatorname{Rep} G \simeq \operatorname{Rep}\mathbb{C}[G] \simeq \operatorname{Rep}(\operatorname{End} U)$$

Now consider and $\alpha \in \operatorname{Aut}^{\otimes} U$

$$\phi\colon \mathbb{C}[\operatorname{Aut}^{\otimes} U] \to \operatorname{End} U$$
$$\alpha \mapsto \alpha$$

This is clearly an injective algebra homomorphism. Thus we conclude

$$|G| = \dim \mathbb{C}[G] = \dim \operatorname{End} U \geq \dim \mathbb{C}[\operatorname{Aut}^{\otimes} U] = |\operatorname{Aut}^{\otimes} U|.$$

It suffices to show $|G| \leq |\operatorname{Aut}^{\otimes} U|$ to see that $\mathbb{C}[\operatorname{Aut}^{\otimes} U] \cong \operatorname{End} U$. Now consider the map

$$\psi\colon G \to \operatorname{Aut}^{\otimes} U$$

where $\psi(g)$ is the natural transformation β with components

$$\psi(g)_{(V,\rho_V)} = \rho_V(g)$$

this is a natural transformation since a G-linear map $f\colon V \to W$ is still G-linear after the forgetful functor. It is also clear that monoidal since we have defined the tensor product of representations to be exactly this. while not strictly necessary for the proof ψ is also a group homomorphism. At last suppose $\psi(g) = \psi(h)$ for $g, h \in G$. Then for all $(V, \rho_V) \in \operatorname{Rep} G$ there is an equality $\rho_V(g) = \rho_V(h)$. In particular this holds for the regular representation on $\mathbb{C}[G]$, where G acts on $\mathbb{C}[G]$ by left multiplication i.e. $\rho_{reg}(g)(x) = gx$. Thus since $\rho_{reg}(g) = \rho_{reg}(h)$ we in particular get that

$$g = ge = \rho_{reg}(g)(e) = \rho_{reg}(h)(e) = he = h.$$

Hence ψ is injective. Therefor $|G| \leq |\operatorname{Aut}^{\otimes} U|$. We conclude that

$$\dim \operatorname{End} U = \dim \mathbb{C}[\operatorname{Aut}^{\otimes} U]$$

hence ϕ is a isomorphism thus

$$\operatorname{Rep}(\operatorname{End} U) \simeq \operatorname{Rep}\mathbb{C}[\operatorname{Aut}^{\otimes} U].$$

Now it follows from Proposition 12.7 that

$$\operatorname{Rep} G \simeq \operatorname{Rep}\mathbb{C}[\operatorname{Aut}^{\otimes} U] \simeq \operatorname{Rep}(\operatorname{Aut}^{\otimes} U).$$

□

Appendix A - String diagrams.

Through out this project we will make use of string diagram formalism. A string diagram is a computational tool that corresponds to a schema of morphisms with parentheses and units applied at will in a monoidal category $\mathscr{C}$ with varying degree of additional structure.

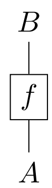
Notation. For a category $\mathscr{C}$ we use the following conventions:

- composition is computed vertically from bottom to top.
- An object $A \in \mathscr{C}$ is denoted by a node

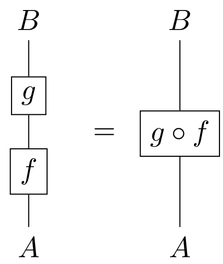
$$A$$

- The identity map id_A is denoted by a string

- a map $f\colon A \to B$ in $\mathscr{C}$ is drawn by adding a label on the string and changing the corresponding codomain node i.e.



  with the convention that maps appropriately composeable

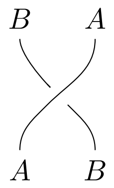


- Assume now that $(\mathscr{C}, \otimes, 1)$ is a monoidal category.
- Two string horizontally next to each other are to be interpreted as the tensor product of maps.
- Maps with the monoidal unit 1 as domain or codomain will be suppressed.
- Natural transformations (applied locally) is expressed as switching the order of labeling.

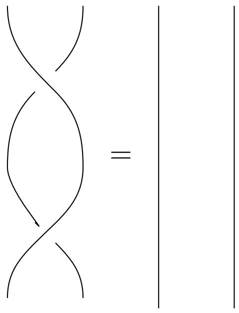
This of course only uses the structure of monoidality. But we will now demonstrate how duals and braidings give additional flexibility in computations with string diagrams.

Notation. From now on we will suppress the object nodes unless if their is no risk of confusion.

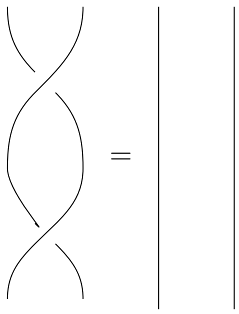
Notation. Let $(\mathscr{C}, \otimes, 1, \beta)$ be a braided monoidal category. For $A, B \in \mathscr{C}$. We denote $\beta_{A,B}$ by



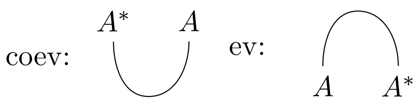
The fact that $\beta^{-1}\beta = \text{id}$ thus corresponds to:



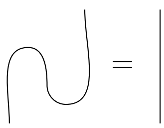
and a symmetric monoidal category satisfies:



Notation. If $(\mathscr{C}, \otimes, 1)$ is a monoidal category and $A \in \mathscr{C}$ and A has right (left) dual then we define the evaluation and coevaluation (the left dual is the mirror image) as:



Making the snake equations the following string diagram:



and its mirror image.

Theorem 12.10 (Joyal and Street)**.** *If* $(\mathscr{C}, \otimes, 1)$ *is a rigid monoidal category, then any evaluation of a string diagram is invariant under planar isotopy.*

The proof of this theorem is out the scope of this project but the theorem is included anyways to ensure the reader of soundness of computations with string diagram. The proof however is included in Joyal and Streets article [JS91].

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