MONADICITY OF THE BOUSFIELD-KUHN FUNCTOR

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These notes are from my talk on "Monadicity of the Bousfield-Kuhn functor." in the course "Topics in Algebraic Topology (2020/2021)" at University of Copenhagen.

The goal of the talk is to construct the Bousfield-Kuhn functor

$$\Phi\colon \mathrm{An}_* \longrightarrow \mathrm{Sp}_{T(n)}$$

and prove that (a modification of the above functor) gives us an monadic adjunction

$$\mathrm{Sp}_{T(n)} \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} \mathrm{An}_*^{v_n}.$$

This witnesses an equivalence

$$\mathrm{An}_*^{v_n} \simeq \mathrm{Alg}_{\Phi\Theta}\left(\mathrm{Sp}_{T(n)}\right)$$

and in fact Heuts prove in [Heu18, Thm. 2.6] that

$$\mathrm{Alg}_{\Phi\Theta}\left(\mathrm{Sp}_{T(n)}\right) \simeq \mathrm{Lie}\left(\mathrm{Sp}_{T(n)}\right)$$

with the right hand side being the ∞-category of $T(n)$-local spectral Lie algebras. For $n = 0$ this recovers Quillens classical result that the rational homotopy theory of simply connected spaces is equivalent to the homotopy theory of rational Lie algebras.

For the entirety of these notes we fix a prime p and let An and Sp denote the ∞-categories of p-local anima and p-local spectra respectively.

1. The ∞-category of v_n-periodic anima

In this section we will define v_n-periodic homotopy groups and give particularly nice model for the Dwyer-Kan localization of the ∞-category of pointed anima An_* at v_n-periodic equivalences.

Definition 1.1. A finite pointed anima $V \in \mathrm{An}_*^{\mathrm{fin}}$ is of type n if the Morava k-theory vanishes $K(m)_*V \simeq 0$ for $m < n$ and $K(n)_*V$ is nonzero. A map v_n-self map is a map $v_n\colon \Sigma^d V \to V$ such that

$$K(m)_* v_n$$

is an isomorphism for $m = n$ and is nilpotent else.

By a theorem of Hopkins and Smith any anima V of type at least n, admits a v_n-self map. For this talk we will fix a sequence of finite pointed suspension anima $(V_n)_{n\in\mathbb{N}}$ with V_n of type n and assume that $V_n \in \tau_{>k}\mathrm{An}_*$ implies $V_{n+1} \in \tau_{>k}\mathrm{An}_*$.

Notation. We consider the afore mentioned sequence of anima and denote by

$$d_n \coloneqq \min(k \in \mathbb{N}_0\colon \pi_k(V_n) \neq 0)$$

the level of the lowest non-zero homotopy group of V_n.

Definition 1.2. Let $X \in \mathrm{An}_*$ be a pointed anima. The v_n-periodic homotopy groups of X are given by the colimit

$$v_n^{-1}\pi_*(X, V) \coloneqq \mathrm{colim}\left(\pi_*\mathrm{Map}_*(V_n, X) \to \pi_*\mathrm{Map}_*(\Sigma^d V_n, X) \to \dots\right).$$

Note that the v_n-periodic homotopy groups are independent, up to unique isomorphism of v_n-self maps. The v_n-periodic homotopy groups of a space X are canonically isomorphic to the stable homotopy groups of a spectrum Φ_{V_n}.

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Definition 1.3. The v_n-telescopic functor $\Phi_{V_n}\colon \mathrm{An}_* \to \mathrm{Sp}$ is the functor given by

$$\Phi_{V_n}(X) := \operatorname{colim}\left(\Sigma^\infty \mathrm{Map}_*(V_n, X) \to \Sigma^{\infty-d}\mathrm{Map}_*(V_n, X) \to \dots\right).$$

A map $f\colon X \to Y$ in An_* is a v_n-periodic equivalences if $\Phi_{V_n}(f)$ is an equivalence.

Remark 1.4. These functors are independent of V_n and v_n-self map up to contractible choice.

These functors will be the focus section two, however for now they will serve to define periodic equivalences. We will now move towards constructing an ∞-category of v_n-periodic anima.

Definition 1.5. Let $L_n^f \tau_{>1}\mathrm{An}_*$ denote the Bousfield localization of simply connected pointed anima $\tau_{>1}\mathrm{An}_*$ at the map $V_{n+1} \to *$ and let

$$L_n^f\colon \tau_{>1}\mathrm{An}_* \to \tau_{>1}\mathrm{An}_*$$

denote the corresponding localization functor.

Remark 1.6. Note that theres a natural equivalence $L_n^f(\tau_{>d_{n+1}}) \simeq \tau_{>d_{n+1}}\left(L_n^f\right)$.

Theorem 1.7 (Theorem 4.6 [Bou01])**.** *If $X \in \mathrm{An}_*$ is a pointed anima, then the v_i-periodic homotopy groups of $L_n^f X$ are given by*

$$v_i^{-1}\pi_*(L_n^f X, V_i) \simeq \begin{cases} v_i^{-1}\pi_*(X, V_i) & i \le n \\ 0 & \textit{else.} \end{cases}$$

Lemma 1.8. *The natural unit transformation $L_{n-1}^f \to L_n^f L_{n-1}^f$ is an equivalence.*

This implies that any L_n^f-local anima is L_{n-1}^f-local. In particular we get a map

$$L_n^f \to L_{n-1}^f.$$

Definition 1.9. Consider the map $L_n^f \to L_{n-1}^f$ and let M_n^f denote fiber.

We choose the letter M to indicate the monochromatic level n. From Bousfields theorem it follows that

$$v_i^{-1}\pi_*(M_n^f X, V_i) \simeq \begin{cases} v_n^{-1}\pi_*(X, V_n) & i = n \\ 0 & \text{else} \end{cases}$$

and choose the convention that $M_0^f = L_0^f$.

Definition 1.10. Let $\mathrm{An}_*^{v_n} \subseteq \tau_{>1}\mathrm{An}_*$, be full subcategory of $\tau_{>1}\mathrm{An}_*$ spanned by anima of the form $\tau_{>d_{n+1}}\left(M_n^f X\right)$. We call this the ∞-category of v_n-periodic anima. We let $i_n\colon \mathrm{An}_*^{v_n} \to \mathrm{An}_*$ denote the inclusion and $M_n\colon \mathrm{An}_* \to \mathrm{An}_*^{v_n}$ the functor given by

$$X \mapsto \tau_{>d_{n+1}}\left(M_n^f X\right).$$

Theorem 1.11 (Theorem 3.7 [Heu18])**.** *A map $\phi\colon X \to Y$ in $\tau_{>d_{n+1}}$ is a v_i-periodic equivalence for all $0 \le i \le n$ if and only if $\tau_{>d_{n+1}}(L_n^f\phi)$ is an equivalence. Furthermore, it is a v_n-periodic equivalence if and only if $\tau_{>d_{n+1}}(M_n^f\phi)$ is an equivalence.*

Lemma 1.12. *There is a natural equivalence $M_n i_n \simeq \mathrm{id}_{\mathrm{An}_*^{v_n}}$.*

Proof. This is clear for $n = 0$. So assume $n > 0$ and let $X = M_n(Y)$ for some $Y \in \mathrm{An}_*$. By definition we have a fiber sequence

$$M_n^f X \to L_n^f X \to L_{n-1}^f X$$

Now, as X is L_n^f-local and d_{n+1}-connected we have that

$$\tau_{>d_{n+1}}(L_n^f X) \simeq L_n^f X \simeq X,$$

so it suffices to prove $L^f_{n-1}X \simeq *$. Now we apply $\tau_{>d_n}(L^f_{n-1})$ to the fiber sequence

$$M^f_n Y \to L^f_n Y \to L^f_{n-1} Y$$

to get a fiber sequence

$$\tau_{>d_n}(L^f_{n-1}(M^f_n Y)) \to \tau_{>d_n}(L^f_{n-1}(L^f_n Y)) \to \tau_{>d_n}(L^f_{n-1}(L^f_{n-1} Y))$$

with the last map being an equivalence. So we have reduced to showing that the map

$$L^f_{n-1}X \simeq \tau_{>d_{n+1}}(L^f_{n-1}(M^f_n Y)) \to \tau_{>d_n}(L^f_{n-1}(M^f_n Y))$$

is an equivalence. By [Heu18, Theorem 3.7] it suffices to show that this is an v_i-periodic equivalence for all $0 \leq i \leq n$. However this is clear as their homotopy groups only vary in finitely many degrees. □

Corollary 1.13. *A map $\phi\colon X \to Y$ in $\mathrm{An}_*^{v_n}$ is an equivalence if and only if it is a v_n-periodic equivalence.*

Proof. This follows from Lemma 1.12 and [Heu18, Theorem 3.7]. □

This is allows us to prove our main theorem for this section.

Theorem 1.14. *The ∞-category $\mathrm{An}_*^{v_n}$ is the Dwyer-Kan localization of An_* at the v_n-periodic equivalences.*

Proof. We have to show that for any ∞-category $\mathcal{C}$ the functor

$$i_n^*\colon \mathrm{Fun}_{v_n}(\mathrm{An}_*, \mathcal{C}) \to \mathrm{Fun}(\mathrm{An}_*^{v_n}, \mathcal{C}),$$

is an equivalence. Here $\mathrm{Fun}_{v_n}(\mathrm{An}_*, \mathcal{C})$ denotes the full subcategory of $\mathrm{Fun}(\mathrm{An}_*, \mathcal{C})$ spanned by functors taking v_n-periodic equivalences to equivalences. Since $i_n^* M_n^* \simeq (M_n i_n)^*$ we see that $i_n^* M_n^* \simeq \mathrm{id}$. Now suppose $F\colon \mathrm{An}_* \to \mathcal{C}$ is a functor which takes v_n-periodic equivalences to equivalences. Now for $X \in \mathrm{An}_*$ consider the following zigzag of natural v_n-periodic equivalences

$$X \leftarrow \tau_{>d_{n+1}}(X) \to \tau_{d_{n+1}}(L^f_n X) \leftarrow iMX.$$

If we apply F to this zigzag, we see that $M^* i^* \simeq \mathrm{id}$. Which is what we wanted to prove. □

Manifestly, the above result depends on choice of d_{n+1}, but this choice is unique up to a contractible space of choices.

2. The Bousfield-Kuhn functor

In this section we will sketch the construction of the Bousfield-Kuhn functor from the telescopic functors and state our main technical ingredient for proving monadicity of the Bousfield-Kuhn functor. The section is based primarily on the Thursday seminar on Unstable chromatic homotopy theory and [Kuh+08].

Recall that if V is a finite pointed anima of type at least n with a v_n-self map $v\colon \Sigma^d V \to V$, then we can construct a functor $\phi_V\colon \mathrm{An}_* \to \mathrm{Sp}$. This functor is in fact functorial in V and v and we use this for our construction.

Construction 2.1 (The Bousfield-Kuhn functor)**.** Let $t > 0$ be a positive integer and let $\mathcal{C}_t$ be the full subcategory of the pullback

$$\begin{array}{ccc} \bar{\mathcal{C}}_t & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathrm{An}_*) \\ \downarrow & & \downarrow{\scriptstyle s\times t} \\ \mathrm{An}_* & \xrightarrow{\Sigma^t \times \mathrm{id}} & \mathrm{An}_* \times \mathrm{An}_* \end{array}$$

spanned by pairs (V, v) where V is of type at least n and $v\colon \Sigma^t V \to V$ is a v_n-self map. From this we obtain a functor

$$\Phi_\bullet\colon \mathcal{C}_t^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{An}_*, \mathrm{Sp})$$

given on objects by $(V, v) \mapsto \Phi_V$. Now for every pair of positive integers t and s there exists a triangle

$$\begin{array}{ccc} \mathcal{C}_t^{\mathrm{op}} & \longrightarrow & \mathcal{C}_{st}^{\mathrm{op}} \\ & {\scriptstyle \Phi_\bullet} \searrow \quad \swarrow {\scriptstyle \Phi_\bullet} & \\ & \mathrm{Fun}(\mathrm{An}_*, \mathrm{Sp}) & \end{array}$$

such that the horizontal functor is the above one. This functor is given objects $(V, v) \mapsto (V, v^s)$ where v^s is the composite

$$\Sigma^{st} V \to \Sigma^{(s-1)t} V \to \cdots \to \Sigma^t V \to V.$$

Furthermore, $(V, v) \mapsto (\Sigma V, \Sigma(v))$ determines a functor such that exists a triangle witnessing that

$$\begin{array}{ccc} \mathcal{C}_t^{\mathrm{op}} & \xrightarrow{\quad \Sigma \quad} & \mathcal{C}_t^{\mathrm{op}} \\ & {\scriptstyle \Phi_\bullet} \searrow \quad \swarrow {\scriptstyle \Omega\Phi_\bullet} & \\ & \mathrm{Fun}(\mathrm{An}_*, \mathrm{Sp}) & \end{array}$$

commutes. Now consider the diagram[1]

$$\mathcal{C}_{1!} \to \mathcal{C}_{2!} \to \ldots$$

where the functor $\mathcal{C}_{(m-1)!} \to \mathcal{C}_{m!}$ is given on objects by $(V, v) \mapsto (\Sigma V, \Sigma(v^m))$. We let $\mathcal{C}'$ denote the colimit of this diagram. By the above we get a functor

$$\mathcal{C}' \to \mathrm{Fun}(\mathrm{An}_*, \mathrm{Sp}).$$

In the Thursday seminar, Lurie proves that

$$\mathcal{C}' \simeq \mathrm{Sp}^{\mathrm{fin}}_{\geq n}$$

where the right hand side is the ∞-category of finite spectra of at least type n. Now let F denote the right Kan extension of the functor $\mathrm{Sp}^{\mathrm{fin}}_{\geq n} \to \mathrm{Fun}(\mathrm{An}_*, \mathrm{Sp})$ which takes $V \in \mathrm{Sp}^{\mathrm{fin}}_{\leq n}$ to Φ_V. Along the inclusion

$$\mathrm{Sp}^{\mathrm{fin}}_{\leq n} \hookrightarrow \mathrm{Sp}^{\mathrm{fin}}.$$

Here $\Phi_V \coloneqq \Sigma^t \Phi_W$ where W is a type n anima such that $\Sigma^t V \simeq \Sigma^\infty W$.

Definition 2.2. Let $\mathcal{C}'$ and

$$F\colon \left(\mathrm{Sp}^{\mathrm{fin}}\right)^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{An}_*, \mathrm{Sp})$$

be as in Construction 2.1. The *Bousfield-Kuhn* functor is the functor

$$\Phi \coloneqq F(\mathbb{S})\colon \mathrm{An}_* \to \mathrm{Sp}.$$

Here $\mathbb{S}$ denotes the sphere spectrum.

Note that since right Kan extensions are given pointwise we obtain a formula for $X \in \mathrm{An}_*$, the Bousfield-Kuhn functor is given on X by

$$\Phi(X) \simeq \lim_{E \to \mathbb{S}} \Phi_E(X)$$

Where the indexing category is the slice category $\mathrm{Sp}^{\mathrm{fin}}_{\geq n}/_{\mathbb{S}}$.

Theorem 2.3 (Thm. 1.1 [Kuh+08])**.** *The Bousfield-Kuhn functor satisfies the following properties*

(1) The functor $\Phi\colon \mathrm{An}_* \to \mathrm{Sp}$ takes values in $T(n)$-local spectra.
(2) If V is a type n anima then for all $X \in \mathrm{An}_*$

$$\Phi(X)^V \simeq \Phi_V(X).$$

(3) There is a natural equivalence

$$\Phi(\Omega^\infty X) \simeq L_{T(n)} X.$$

[1]Note that the spine inclusion $I \to \Delta^n$ into the n-simplex is Joyal equivalence. So we really do get a diagram in Cat_∞

The above theorem will work as our major technical result for the rest these notes.

Remark 2.4. Note that Theorem 2.3 (3) tells us that the $T(n)$-localization only depends on the underlying anima!

Lemma 2.5 (Thursdays seminar lecture 5 Prop. 13)**.** *The v_n-periodic telescopic functor*

$$\Phi_V\colon \mathrm{An}_* \to \mathrm{Sp}$$

takes values in $T(n)$-local spectra.

Lemma 2.6. *The v_n-periodic telescopic functor $\Phi_V\colon \mathrm{An}_* \to \mathrm{Sp}_{T(n)}$ factors through*

$$M\colon \mathrm{An}_* \to \mathrm{An}_*^{v_n}.$$

Proof. This follows from applying the universal property of Dwyer-Kan localization. □

Corollary 2.7. *The Bousfield-Kuhn functor $\Phi\colon \mathrm{An}_* \to \mathrm{Sp}_{T(n)}$ factors through*

$$M\colon \mathrm{An}_* \to \mathrm{An}_*^{v_n}.$$

Proof. This follows from applying Theorem 2.3.2 and Lemma 2.6. □

Notation. We will abuse notation and denote the functor

$$\mathrm{An}_*^{v_n} \to \mathrm{Sp}_{T(n)}$$

from Corollary 2.7 by Φ and call it the Bousfield-Kuhn functor.

Theorem 2.8. *The Bousfield-Kuhn functor $\Phi\colon \mathrm{An}_*^{v_n} \to \mathrm{Sp}_{T(n)}$ admits a left adjoint*

$$\Theta\colon \mathrm{Sp}_{T(n)} \to \mathrm{An}_*^{v_n}.$$

Proof. We want to show that the functor

$$X \mapsto \mathrm{Map}(Y, \Phi(X))$$

is representable. Now as $\Phi(X) \simeq \lim_{E\to\mathbb{S}} \Phi_E(X)$ it suffices to show that the functor

$$X \mapsto \mathrm{Map}(Y, \Phi_V(X))$$

is representable, where V is a type n anima with v_n-self map $\Sigma^t V \to V$. Now as Sp is generated under colimits by $\Sigma^{kt}\mathbb{S}$ for $k \in \mathbb{Z}$ it suffices to show that the functor

$$X \mapsto \mathrm{Map}(\Sigma^{kt}\mathbb{S}, \Phi_V(X))$$

is representable. Now since $\Phi_V(X) \simeq \Omega^t\Phi_V(X)$ this reduces to showing that

$$X \mapsto \mathrm{Map}(\mathbb{S}, \Phi_V(X)) \simeq \Omega^\infty\Phi_V(X)$$

is representable. So we are reduced to prove that the functor

$$X \mapsto \mathrm{colim}(\mathrm{Map}(V, X) \to \mathrm{Map}(\Sigma^t V, X) \to \dots)$$

is representable as a functor $\mathrm{An}_*^{v_n} \to \mathrm{An}_*$ is representable. Now since $X \in \mathrm{An}_*^{v_n}$ the diagram above is eventually constant so it suffices to prove $X \mapsto \mathrm{Map}(\Sigma^{kt}V, X)$ is representable. This we know to be representable by $L_n^f(\Sigma^{kt}V)$. This completes the proof. □

3. Monadicity of the Bousfield-Kuhn functor

In this section we prove the main theorem of these notes, namely that the Bousfield-Kuhn functor witnesses that $\mathrm{An}_*^{v_n}$ is mondadic over $\mathrm{Sp}_{T(n)}$. First we recall the Barr-Beck-Lurie theorem.

Theorem 3.1 (Thm. 4.7.0.3 [Lur17])**.** *Suppose a functor $G\colon \mathcal{D} \to \mathcal{C}$ admits a left adjoint $F\colon \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ admits geometric realizations of simplicial objects. Suppose the following are satisfied*

(1) the functor G is conservative,
(2) the functor G preserves geometric realizations of simplicial objects.

In this situation $\mathcal{D}$ is monadic over $\mathcal{C}$ via the monad GF.

Following the strategy provided by the Barr-Beck-Lurie theorem and Theorem 2.8 we will now be able to prove our main theorem.

Theorem 3.2. *The Bousfield-Kuhn functor witnesses that* $\mathrm{An}_*^{v_n}$ *is monadic over* $\mathrm{Sp}_{T(n)}$.

Proof. We prove the two assumptions in the Barr-Beck-Lurie theorem Theorem 3.1 are satisfied when $\Phi = G$ and $\Theta = F$.

For (1) if $f\colon X \to Y$ is a map in $\mathrm{An}_*^{v_n}$ such that $\Phi(f)$ is an equivalence, then $\Phi(f)^V$ is an equivalence. So by naturality we get that $\Phi_V(f)$ is an equivalence. So f is an v_n-periodic equivalence.

For (2) let $X_\bullet \in (\mathrm{An}_*^{v_n})^{\Delta^{\mathrm{op}}}$ be a simplicial object in $\mathrm{An}_*^{v_n}$. We wish to show that the canonical map

$$|\Phi(X_\bullet)| \longrightarrow \Phi(|X_\bullet|)$$

is an equivalence. Now since Φ takes value in $T(n)$-local spectra it suffices to show that

$$|\Phi(X_\bullet)|^V \to \Phi(|X_\bullet|)^V \simeq \Phi_V(|X_\bullet|)$$

is an equivalence for any finite type n anima V. By assumption V is finite so we have equivalences

$$|\Phi(X_\bullet)|^V \simeq |\Phi(X_\bullet)^V| \simeq |\Phi_V(X_\bullet)|.$$

From this we are reduced to showing that $\Phi_V\colon \mathrm{An}_*^{v_n} \to \mathrm{Sp}_{T(n)}$ preserves geometric realizations. By [Heu18, Lemma 3.17] the inclusion

$$\mathrm{An}_*^{v_n} \hookrightarrow L_n^f \mathrm{An}_*$$

preserves colimits and colimits in $L_n^f \tau_{>d_{n+1}} \mathrm{An}_*$ are computed by the formula

$$\operatorname*{colim}_I F \simeq L_n^f \left(\operatorname*{colim}_I iF \right)$$

From this we conclude that $|X_\bullet| \simeq L_n^f(|i(X_\bullet)|)$. Now the canonical transformation $\mathrm{id} \to L_n^f$ is a v_n-periodic equivalence so it suffices to show

$$\Phi_V\colon \tau_{>d_{n+1}} \mathrm{An}_* \to \mathrm{Sp}_{T(n)}$$

preserves geometric realizations. Now as Σ^∞ preserves colimits it suffices to show that

$$\mathrm{Map}(V_n, -)\colon \tau_{>d_{n+1}} \mathrm{An}_* \to \mathrm{An}$$

preserves geometric realizations. This now follows by induction on skeleton of V_n and the assumption that $conn(V_n) \leq conn(V_{n+1})$. A proof is also given in [Eld+19, Prop 4.2].

As desired we conclude from the Barr-Beck-Lurie theorem that

$$\mathrm{An}_*^{v_n} \simeq \mathrm{Alg}_{\Phi\Theta}(\mathrm{Sp}_{T(n)}).$$

□

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