INTRODUCTION TO STABLE ∞-CATEGORIES

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These notes are from my talk on "Introduction to Stable ∞-categories." in the course "Topics in Algebraic Topology (2020/2021)" at University of Copenhagen.

1. Foundations

We will spend this section on defining stable ∞-categories and prove basic results about these.

Recall that an object $x \in \mathcal{C}$ is final if $\mathrm{Map}_{\mathcal{C}}(y, x) \in pt$ for all $y \in \mathcal{C}$ and that it is initial if it is final in $\mathcal{C}^{op}$.

Definition 1.1. Let $\mathcal{C}$ be an ∞-category. An object $0 \in \mathcal{C}$ is a *zero* object if it is both initial and final. In this case we say $\mathcal{C}$ is pointed.

Remark 1.2. This implies that there is a distinguished element

$$X \to 0 \to Y$$

in $\hom_{h\mathcal{C}}(X, Y)$ for all $X, Y \in \mathcal{C}$.

Definition 1.3. Let $\mathcal{C}$ be a pointed ∞-category. A functor $\sigma \in \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ is an *triangle* if it is of the form

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

where 0 is a zero object of $\mathcal{C}$. The triangle σ is a *fiber sequence* if it is a pullback square, *cofiber sequence* if it is a pushout square and *bifiber sequence* if it is both. We say that $\mathcal{C}$ admits (co)fibers if any map $f\colon X \to Y$ in $\mathcal{C}$ has a (co)fiber.

We note that a triangle comes with the information of three objects A, B and C, to composeable maps $f\colon A \to B$ and $g\colon B \to C$ and a nullhomotopy $gf \simeq 0$, where 0 denotes the zero map from A to C.

Notation. We will often abuse notation and say that

$$X \to Y \to Z$$

is a triangle/fiber/cofiber sequence.

Definition 1.4. An ∞-category $\mathcal{C}$ is *stable* if

(1) the ∞-category $\mathcal{C}$ is pointed.
(2) the ∞-category $\mathcal{C}$ admits both fibers and cofibers.
(3) A triangle $X \to Y \to Z$ is a fiber sequence if and only if it is a cofiber sequence.

We will now prove basic properties of stable ∞-categories.

Theorem 1.5. *If $\mathcal{C}$ is a stable ∞-category, then $\mathcal{C}$ admits finite limits and colimits.*

Proof. Note that by [Lur09, Corollary 4.4.2.4] since $\mathcal{C}$ is pointed it suffices to show that finite pullbacks and pushouts exists. We consider a diagram

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$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle f} & & \\ Y & & \end{array}$$

first form the fiber of f to get a diagram

$$\begin{array}{ccccc} \mathrm{fib}(f) & \xrightarrow{i} & X & \xrightarrow{g} & Z \\ \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & Y & & \end{array}$$

and then form the cofiber of gi, to get a diagram

$$\begin{array}{ccccc} \mathrm{fib}(f) & \xrightarrow{i} & X & \xrightarrow{g} & Z \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ 0 & \longrightarrow & Y & \longrightarrow & \mathrm{cofib}(gi). \end{array}$$

In this diagram the outer square is a pushout, and since $\mathcal{C}$ is stable, so is the left square. Hence by [Lur09, Lemma 4.4.2.1] it follows that the left square is a pushout. One similarly shows that finite pullbacks exists. □

Theorem 1.6. *If $\mathcal{C}$ is a pointed ∞-category which admits fibers and cofibers, then there exists adjoint functors*

$$\Sigma\colon \mathcal{C} \rightleftarrows \mathcal{C} \colon \Omega$$

which on objects are given by $\Sigma(X) = \mathrm{cofib}(X \to 0)$ and $\Omega(X) = \mathrm{fib}(0 \to X)$. Furthermore if $\mathcal{C}$ is stable then these functors are inverse equivalences.

Proof. We will only produce the functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$. The loop functor is produced in an analogous way.

Let $M^{\Sigma} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ denote the full subcategory spanned by pushout diagrams of the form

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & Z, \end{array}$$

where 0 and $0'$ are zero objects in $\mathcal{C}$. Now consider $\Lambda^2_0 \subseteq \Delta^1 \times \Delta^1$ and let $L^{\Sigma} \subseteq \mathrm{Fun}(\Lambda^2_0, \mathcal{C})$ be the full subcategory spanned by diagrams of the form

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \\ 0 & & . \end{array}$$

The restriction functor $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Lambda^2_0, \mathcal{C})$ restricts to a functor $M^{\Sigma} \to L^{\Sigma}$, [Lur09, Prop. 4.3.2.15] shows that this map is a trivial Kan fibration. Similarly it follows from [Lur09, Prop. 4.3.2.15] the map $L^{\Sigma} \to \mathcal{C}$ given by evaluation at the initial vertex is trivial Kan fibration. So we get a trivial Kan fibration

$$M^{\Sigma} \to L^{\Sigma} \to \mathcal{C}$$

which admits a section $s\colon \mathcal{C} \to M^{\Sigma}$. There for we can construct sigma to be the composite

$$\mathcal{C} \xrightarrow{s} M^{\Sigma} \xrightarrow{i} \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \xrightarrow{ev_{(1,1)}} \mathcal{C}$$

where i is the inclusion and $ev_{(1,1)}$ is evaluation at the final vertex of $\Delta^1 \times \Delta^1$.

We will not show that the functors Σ and Ω are adjoint, but we will construct the unit and counit of the adjunction. We note that since the mapping space functors are left exact (in fact the preserve all limits) it follows that

$$\mathrm{Map}_{\mathcal{C}}(\Sigma X, Y) \simeq \Omega_{\mathrm{An}}\mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \mathrm{Map}_{\mathcal{C}}(X, \Omega Y).$$

Furthermore, forming the suspension of X we get that

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array}$$

is a pushout diagram, and thus by commutativity we get a map $X \xrightarrow{\eta_X} \Omega\Sigma X$, similarly we obtain a map $\Sigma\Omega X \xrightarrow{\varepsilon_X} X$. These form the counit and unit of adjunction this is [Lur17, Remark 1.1.2.8].

Finally if $\mathcal{C}$ is stable, then η and ε are equivalences, so Σ and Ω are mutual inverses. □

Theorem 1.7 ([Lur17], Thm. 1.4.2.11). *Let $\mathcal{C}$ be a pointed ∞-category which admits finite limits and colimits. Then*

1. *If Σ is fully faithful, then every pushout square is a pullback square.*
2. *If Ω is fully faithful, then every pullback square is a pushout square.*
3. *If Ω is an equivalence, then $\mathcal{C}$ is stable.*
4. *If Σ is an equivalence, then $\mathcal{C}$ is stable.*

Definition 1.8. If $\mathcal{C}$ and $\mathcal{D}$ are stable ∞-categories, then a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *exact* if it preserves fiber sequences

Recall that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is left exact if it commutes with finite limits. Likewise, it is left exact if it commutes with finite colimits.

Notation. If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories we denote by

$$\mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}, \mathcal{D})$$

the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by left exact functors. Similarly we denote by $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{C}, \mathcal{D})$ the full subcategory spanned by right exact functors.

Proposition 1.9 ([Lur17], Prop. 1.1.4.1). *If $\mathcal{C}$ and $\mathcal{D}$ are stable ∞-categories and $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, then the following are equivalent*

1. *the functor F is exact.*
2. *The functor F is left exact, that is commutes with finite limits.*
3. *The functor F is right exact, that is commutes with finite colimits.*

2. Stabilization

The goal for this section is to describe stabilization a procedure which takes in an ∞-category and produces a stable ∞-category.

Definition 2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then we say that

1. the functor F is reduced if it preserves final objects.
2. The functor F is excisive it it takes pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$.

Notation. We let $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$, $\mathrm{Exc}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ denote the full subcategories of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by reduced, excisive and excisive reduced functors respectively.

Lemma 2.2. *If $\mathcal{C}$ is a small pointed ∞-category with finite colimits and $\mathcal{D}$ is an ∞-category with finite limits. Then $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ is pointed and admits finite limits.*

Proof. Since $\mathcal{D}$ has finite limits, it follows by [Lur09, Cor. 5.1.2.3] that $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ also has. Now $\mathrm{Exc}_*(\mathcal{C},\mathcal{D})$ is closed under limits in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ and the inclusion

$$\mathrm{Exc}_*(\mathcal{C},\mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C},\mathcal{D})$$

is fully faithfull, and thus reflects limits, it follows that $\mathrm{Exc}_*(\mathcal{C},\mathcal{D})$ has finite limits.

It remains to show that $\mathrm{Exc}_*(\mathcal{C},\mathcal{D})$ is pointed. We consider $X\colon \mathcal{C}\to\mathcal{D}$ which is constant on the final object $*\in\mathcal{D}$. Clearly X is final, so it remains to show that it is initial. To do this we will show that it is initial in the ∞-category of reduced functors from $\mathcal{C}$ to $\mathcal{D}$. Let $Y\in\mathrm{Fun}_*(\mathcal{C},\mathcal{D})$. Since $\mathcal{C}$ is pointed, $\mathcal{D}$ has $\mathcal{C}$-indexed limits which are computed by evaluating functors on a zero object $0\in\mathcal{C}$. It follows that

$$\begin{aligned}\mathrm{Map}(X,Y) &\simeq \mathrm{Map}_{\mathcal{D}}(*,\lim_{\mathcal{C}} Y)\\ &\simeq \mathrm{Map}_{\mathcal{D}}(*,Y(0))\\ &\simeq \mathrm{Map}_{\mathcal{D}}(*,*)\\ &\simeq pt.\end{aligned}$$

Here the second to last equivalence follows from Y being reduced. So it follows that X initial in $\mathrm{Fun}_*(\mathcal{C},\mathcal{D})$ and hence in $\mathrm{Exc}_*(\mathcal{C},\mathcal{D})$. □

Theorem 2.3. *If $\mathcal{C}$ is a small pointed ∞-category with finite colimits and if $\mathcal{D}$ is an ∞-category with finite limits, then $\mathrm{Exc}_*(\mathcal{C},\mathcal{D})$ is stable.*

Proof. By Theorem 1.7 and Lemma 2.2, it suffices to prove that $\Omega\colon \mathrm{Exc}_*(\mathcal{C},\mathcal{D})\to\mathrm{Exc}_*(\mathcal{C},\mathcal{D})$ is an equivalence. Consider the functor $S\colon \mathrm{Fun}(\mathcal{C},\mathcal{D})\to\mathrm{Fun}(\mathcal{C},\mathcal{D})$ given by precomposition with $\Sigma\colon\mathcal{C}\to\mathcal{C}$. It restricts to an endofunctor on $\mathrm{Exc}_*(\mathcal{C},\mathcal{D})$. We let $F\in\mathrm{Exc}_*(\mathcal{C},\mathcal{D})$ and consider the canonical map

$$F(X)\to\Omega F(\Sigma X).$$

Since F is excisive and reduced this map is an equivalence for all $X\in\mathcal{C}$. Hence by [Lan20, Theorem 8.6] Ω_E is an equivalence, so $\mathrm{Exc}_*(\mathcal{C},\mathcal{D})$ is stable. □

Definition 2.4. Let An denote the ∞-category of *anima* and let An_* denote the ∞-category of pointed anima. We define the ∞-category of *finite anima* $\mathrm{An}^{\mathrm{fin}}$ to the smallest full subcategory of An closed under finite colimits containing the point pt and $\mathrm{An}^{\mathrm{fin}}_*$ denote the ∞-category of *finite pointed anima*.

Remark 2.5. The ∞-category of finite anima $\mathrm{An}^{\mathrm{fin}}$ is essentially small and has the following universal property. If $\mathcal{D}$ has finite colimits then

$$\mathrm{Fun}^{\mathrm{Rex}}(\mathrm{An}^{\mathrm{fin}},\mathcal{D})\xrightarrow{ev_{pt}}\mathcal{D}$$

is an equivalence. This follows from [Lur09, Remark 5.3.5.9] and [Lur09, Prop. 4.3.2.15]

Definition 2.6. If $\mathcal{C}$ be an ∞-category with finite limits. Then the ∞-category of *spectrum objects* in $\mathcal{C}$ is $\mathrm{Exc}_*(\mathrm{An}^{\mathrm{fin}}_*,\mathcal{C})$.

Definition 2.7. The ∞-category of *spectra* Sp is $\mathrm{Exc}_*(\mathrm{An}^{\mathrm{fin}}_*,\mathrm{An})$.

Corollary 2.8. *If $\mathcal{C}$ be an ∞-category with finite limits, then the ∞-category $\mathrm{Sp}(\mathcal{C})$ is stable.*

Definition 2.9. Let $\mathcal{C}$ be an ∞-category with finite limits, then the *infinite delooping functor* is the functor $\Omega^\infty\colon\mathrm{Sp}(\mathcal{C})\to\mathcal{C}$ given by evaluation at S^0.

Theorem 2.10. *Let $\mathcal{C}$ be an ∞-category with finite limits, then the following are equivalent*

(1) $\mathcal{C}$ is stable,
(2) $\Omega^\infty\colon\mathrm{Sp}(\mathcal{C})\to\mathcal{C}$.

Proof. Clearly (2) implies (1). Assume that $\mathcal{C}$ is stable and let $f\colon\mathrm{An}^{\mathrm{fin}}\to\mathrm{An}^{\mathrm{fin}}_*$ be left adjoint to the forgetful functor $U\colon\mathrm{An}^{\mathrm{fin}}_*\to\mathrm{An}^{\mathrm{fin}}$ given by adding a disjoint base point. Furthermore let $\mathrm{Exc}'(\mathrm{An}^{\mathrm{fin}}, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{An}^{\mathrm{fin}}, \mathcal{C})$ spanned by the excisive functors which carry the point *pt* to a final object in $\mathcal{C}$. By [Lur17, Lemma 1.4.2.19] the functor

$$\mathrm{Sp}(\mathcal{C}) \xrightarrow{f^*} \mathrm{Exc}'(\mathrm{An}^{\mathrm{fin}}, \mathcal{C})$$

Is an equivalence. Furthermore, by Remark 2.5 we get that evaluation at a point is an equivalence, so we get a commutative triangle

$$\begin{array}{ccc} \mathrm{Sp}(\mathcal{C}) & \xrightarrow{f^*} & \mathrm{Exc}'(\mathrm{An}^{\mathrm{fin}}, \mathcal{C}) \\ & {}_{\Omega^\infty}\searrow \quad \swarrow_{ev_*} & \\ & \mathcal{C} & \end{array}$$

in which two of the three are equivalences, so $\Omega^\infty \colon \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ is an equivalence. □

Corollary 2.11. *If $\mathcal{C}$ is a small pointed ∞-category with finite colimits and if $\mathcal{D}$ is an ∞-category with finite limits, then $\Omega^\infty \colon \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ induces an equivalence*

$$\mathrm{Exc}_*(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \xrightarrow{(\Omega^\infty)_*} \mathrm{Exc}_*(\mathcal{C}, \mathcal{D}).$$

Proof. This follows from verifying that

$$\mathrm{Exc}_*(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \simeq \mathrm{Sp}(\mathrm{Exc}_*(\mathcal{C}, \mathcal{D}))$$

and applying the previous theorem. □

Corollary 2.12. *If $\mathcal{C}$ is a small stable ∞-category and $\mathcal{D}$ is an ∞-category with finite limits, then $\Omega^\infty \colon \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ induces an equivalence*

$$\mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \xrightarrow{(\Omega^\infty)_*} \mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}, \mathcal{D}).$$

Proof. If $\mathcal{C}$ is stable and $\mathcal{D}$ has finite limits then

$$\mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Exc}_*(\mathcal{C}, \mathcal{D}).$$

So the result follows from the above result. □

This allows us to prove that stable ∞-categories are enriched in spectra in the following sense

Theorem 2.13. *Let $\mathcal{C}$ be a small stable ∞-category. For every $X \in \mathcal{C}$ there exists a functor $\mathrm{map}(X, -) \colon \mathcal{C} \to \mathrm{Sp}$ such that*

$$\Omega^\infty \mathrm{map}(X, -) \simeq \mathrm{Map}(X, -).$$

Proof. We apply Corollary 2.12 to $\mathrm{Map}(X, -)$. □

The final theorem of this section will be to identify $\mathrm{Sp}(\mathcal{C})$ with sequential spectra in in the following sense

Theorem 2.14. *Let $\mathcal{C}$ be a pointed ∞-category with finite limits, then $\Omega^\infty \colon \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ induces an equivalence*

$$\mathrm{Sp}(\mathcal{C}) \xrightarrow{\sim} \lim \left(\dots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \right).$$

Proof. Let $\bar{\mathcal{C}} := \lim \left(\dots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \right)$. It is easy to see that $\bar{\mathcal{C}}$ is stable. The inclusion $\mathrm{Cat}_\infty^{\mathrm{Lex}} \subseteq \mathrm{Cat}_\infty$ preserves limits so the canonical map $G \colon \bar{\mathcal{C}} \to \mathcal{C}$ is left exact. Therefore, it factors as

$$\begin{array}{ccc} \bar{\mathcal{C}} & \xrightarrow{G} & \mathcal{C} \\ & {}_{G'}\searrow \quad \nearrow_{\Omega^\infty} & \\ & \mathrm{Sp}(\mathcal{C}). & \end{array}$$

Let $\mathcal{D}$ be a small stable ∞-category, then Corollary 2.12 implies that the functor

$$\mathrm{Fun}^{\mathrm{Lex}}(\mathcal{D}, \mathrm{Sp}(\mathcal{C})) \xrightarrow{(\Omega^\infty)_*} \mathrm{Fun}^{\mathrm{Lex}}(\mathcal{D}, \mathcal{C})$$

is an equivalence. So it suffices to show that $\Omega\colon \mathcal{C} \to \mathcal{C}$ induces an equivalence on $\mathrm{Fun}^{\mathrm{Lex}}(\mathcal{D}, \mathcal{C})$. Now, $\mathcal{D}$ is stable so the conclusion follows. □

Corollary 2.15. *If $\mathcal{C}$ is presentable, then $\mathrm{Sp}(\mathcal{C})$ is an presentable ∞-category.*

Proof. By [Lur09, Thm. 5.5.3.18], the inclusion $\mathrm{Pr}^R \subseteq \widehat{\mathrm{Cat}_\infty}$ preserves limits, so $\bar{\mathcal{C}}$ is presentable. □

In particular this implies that the ∞-category of spectra Sp is presentable. In fact when restricting to presentable ∞-categories we get the following result.

Theorem 2.16 ([Lur17], Prop. 1.4.4.4 and Cor. 1.4.4.5)**.** *If $\mathcal{C}$ is an presentable ∞-category and $\mathcal{D}$ is a presentable stable ∞-category, then $\Omega^\infty\colon \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\Sigma^\infty_+\colon \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ which induces an equivalence*

$$\mathrm{LFun}(\mathcal{C}, \mathcal{D}) \xrightarrow{\Sigma^\infty_+} \mathrm{LFun}(\mathrm{Sp}(\mathcal{C}), \mathcal{D}).$$

where $\mathrm{LFun}(-,-)$ denotes the full subcategory of $\mathrm{Fun}(-,-)$ spanned by left adjoint functors.

Definition 2.17. Let An be the ∞-category of anima and consider the ∞-category of spectra Sp. The *sphere spectrum*, $\mathbb{S}$, is the image of $pt \in \mathrm{An}$ under $\Sigma^\infty_+\colon \mathrm{An} \to \mathrm{Sp}$.

3. Prestable ∞-categories

In this section we will introduce prestable ∞-categories. Under mild conditions prestable ∞-categories can always be seen as the connective part of stable ∞-category.

Definition 3.1. Let $\mathcal{C}$ be a pointed ∞-category with finite colimits, then the Spanier-Whitehead category on $\mathcal{C}$ is the colimit

$$SW(\mathcal{C}) \coloneqq \mathrm{colim}\left(\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \dots\right).$$

We will study this ∞-category a little before moving on. First note that if I^n denotes the n'th spine, then the inclusion $I^n \to \Delta^n$ is inner anodyne. This is seen in the proof of [Lur09, Prop. 3.2.1.13]. Now the class of inner anodyne maps is saturated so passing to colimits gives an inner anodyne map

$$I^\infty \to \mathbb{N}.$$

Now by [Lur09, Lem. 2.2.5.2] any inner anodyne map is in particular a Joyal equivalence, so defining a functor $\mathbb{N} \to \mathrm{Cat}_\infty$ is equivalent to giving a sequence of ∞-categories and functors between consecutive ones. In particular by unstraigthening this functor there is a model for $SW(\mathcal{C})$, where the objects are pairs (X, n) where $X \in \mathcal{C}$ and $n \in \mathbb{Z}$. Furthermore, using that Δ^1 is compact one can see that the mapping spaces in $SW(\mathcal{C})$ is given by

$$\mathrm{Map}_{SW(\mathcal{C})}((X,n),(Y,m)) \simeq \underset{k}{\mathrm{colim}}\, \mathrm{Map}(\Sigma^{k+n}X, \Sigma^{k+m}Y).$$

Note also that the inclusion $\mathrm{Cat}^{\mathrm{Rex}}_\infty \subseteq \mathrm{Cat}_\infty$ preserves colimits, in fact it is closed under colimits in Cat_∞. So the we can compute the colimit in both, and the answer will agree. This also implies that the canonical map $p\colon \mathcal{C} \to SW(\mathcal{C})$ preserves colimits.

The Spanier-Whitehead category on $\mathcal{C}$ is similar to the spectrum objects $\mathrm{Sp}(\mathcal{C})$ in $\mathcal{C}$ in the following sense.

Theorem 3.2 (Prop. C.1.1.7 [Lur18])**.** *Let $\mathcal{C}$ be a pointed ∞-category with finite colimits, then*

(1) the ∞-category $SW(\mathcal{C})$ is stable.

(2) For any stable ∞-category $\mathcal{D}$ precomposition with the canonical functor $p\colon \mathcal{C} \to SW(\mathcal{C})$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{Rex}}(SW(\mathcal{C}), \mathcal{D}) \xrightarrow{p^*} \mathrm{Fun}^{\mathrm{Rex}}(\mathcal{C}, \mathcal{D}).$$

We will now define prestable ∞-categories.

Definition 3.3. Let $\mathcal{C}$ be an ∞-category, then we say that $\mathcal{C}$ is *prestable* if

(1) $\mathcal{C}$ is pointed and admits finite colimits.
(2) The suspension functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ is fully faithful.
(3) For every map $Z \to \Sigma X$ there exits an bifiber sequence
$$Y \to Z \to \Sigma X.$$

Example 3.4. If $\mathcal{C}$ be a stable ∞-category with t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$, then $\mathcal{C}_{\geq 0}$ is prestable. Indeed:

(1) This is clear as $\mathcal{C}_{\geq 0}$ is a colocalization of $\mathcal{C}$.
(2) This follows from the fact that
$$\begin{array}{ccc} \mathcal{C}_{\geq 0} & \hookrightarrow & \mathcal{C} \\ \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \Sigma} \\ \mathcal{C}_{\geq 0} & \hookrightarrow & \mathcal{C} \end{array}$$
commutes.
(3) This is equivalent to $\mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}$ be closed under extensions, which is an easy check.

Theorem 3.5. *If $\mathcal{C}$ be an ∞-category, then the following are equivalent*

(1) $\mathcal{C}$ is prestable and admits finite limits.
(2) $\mathcal{C}$ is pointed and admits finite colimits, the canonical map $p\colon \mathcal{C} \to SW(\mathcal{C})$ is fully faithful. Moreover, $SW(\mathcal{C})$ admits a t-structure $(SW(\mathcal{C})_{\geq 0}, SW(\mathcal{C})_{\leq 0})$ such that $SW(\mathcal{C})_{\geq 0}$ is the essential image of p.
(3) There exists a stable ∞-category $\mathcal{D}$ with t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ and an equivalence $\mathcal{C} \simeq \mathcal{D}_{\geq 0}$.

Proof. Note that (2) implies (3) is trivial and (3) implies (1) is the above example, so it suffices to prove (1) implies (2).

It is clear from the earlier remark on mapping spaces that p is fully faithful. Now let $SW(\mathcal{C})_{\geq 0}$ be the essential image of p and $SW(\mathcal{C})_{\leq 0}$ be the full subcategory spanned by objects of the form $\Omega^n p(c)$ where $c \in \mathcal{C}$ is n-truncated. In the sense that is $\mathrm{Map}(d, c)$ is n-truncated for all $d \in \mathcal{C}$. We will show that this is a t-structure on $SW(\mathcal{C})$. We verify (i)-(iii) in [Lur17, Def. 1.2.1.1].

(i) Let $X \in SW(\mathcal{C})_{\geq 0}$ and $Y \in SW(\mathcal{C})_{\leq 0}$ then $X \simeq p(c)$ for some $c \in \mathcal{C}$ and $Y \simeq \Omega^n p(c')$ for some n-truncated $c' \in \mathcal{C}$. Therefore the following holds
$$\begin{aligned} \mathrm{Map}_{SW(\mathcal{C})}(X, \Omega Y) &\simeq \Omega^{n+1}\mathrm{Map}_{SW(\mathcal{C})}(p(c), p(c')) \\ &\simeq \Omega^{n+1}\mathrm{Map}_{\mathcal{C}}(c, c') \\ &\simeq pt. \end{aligned}$$
(ii) $\Sigma SW(\mathcal{C})_{\geq 0} \subseteq SW(\mathcal{C})_{\geq 0}$ since $\Sigma p(c) \simeq p(\Sigma c)$ for all $c \in \mathcal{C}$. To see that $\Omega SW(\mathcal{C})_{\leq 0} \subseteq SW(\mathcal{C})_{\leq 0}$, note that n-truncated object is in particular $(n+1)$-truncated.
(iii) We need to show that for any $X \in SW(\mathcal{C})$ there exist a fiber sequence
$$X' \to X \to X''$$
with $X' \in SW(\mathcal{C})_{\geq 0}$ and $X'' \in SW(\mathcal{C})_{\leq 0}$. Note that $X \simeq \Omega^n p(c)$, for some $c \in \mathcal{C}$ and consider the cofiber sequence
$$\Sigma^n \Omega^n c \to c \xrightarrow{\beta} c''.$$
Since p preserves cofiber sequences,
$$\Omega^n p(\Sigma^n \Omega^n c) \to \Omega^n p(c) \xrightarrow{\Omega^n p(\beta)} \Omega^n p(c'')$$
is a cofiber sequence. But
$$\Omega^n p(\Sigma^n \Omega^n c) \simeq \Omega^n \Sigma^n p(\Omega^n c) \simeq p(\Omega^n c)$$

so the left hand term in the cofiber sequence is in $SW(\mathcal{C})_{\geq 0}$. Thus it suffices to show that c'' is $(n-1)$-truncated. We must show that for all $d \in \mathcal{C}$ any map $u\colon \Sigma^m d \to c''$ with $m \geq n$ is nullhomotopic. We consider the following diagram

$$\begin{array}{ccccccc} \mathrm{fib}(\alpha) & \longrightarrow & Y & \longrightarrow & \Omega^n p(c) & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \Omega^n p(\beta)} & & \downarrow \\ 0 & \longrightarrow & \Omega^n p(\Sigma^m d) & \xrightarrow{\Omega^n p(u)} & \Omega^n p(c'') & \longrightarrow & \mathrm{cofib}(\Omega^n p(\beta)). \end{array}$$

All squares are bicartesian, so in particular,

$$\mathrm{fib}(\alpha) \simeq \Omega(\mathrm{cofib}(\Omega^n p(\beta))) \simeq \Omega^{n+1} p(\mathrm{cofib}\beta).$$

By rotating $\mathrm{cofib}\beta \simeq \Sigma^{n+1}\Omega^n c$, we get that

$$\mathrm{fib}(\alpha) \simeq \Omega^{n+1} p(\mathrm{cofib}\beta) \simeq \Omega^{n+1} p(\Sigma^{n+1}\Omega^n c) \simeq p(\Omega^n c).$$

Now since p is closed under extensions and $m \geq n$ we get that $Y \simeq p(E)$ for some $E \in \mathcal{C}$. Using adjointness of Σ^n and Ω^n and fully faithfulness of p, $\Sigma^n E$ into a commutative diagram

$$\begin{array}{ccccc} \Sigma^n\Omega^n c & \longrightarrow & \Sigma^n E & \longrightarrow & \Sigma^m d \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle u} \\ \Sigma^n\Omega^n c & \longrightarrow & c & \xrightarrow{\beta} & c''. \end{array}$$

Again using adjointness we see that u factors through

$$\Sigma^n\Omega^n c \to c \xrightarrow{\beta} c''$$

so it is nullhomotopic. This completes the proof.

□

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