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Chapter 1

Summary

This thesis broadly focuses on studying stable ∞-categories via geometric methods. Specifically, this is done via algebraic K-theory, the Balmer spectrum of a tensor-triangulated category and spectral algebraic geometry. This comes to fruition in three papers, each of which focuses on one of these methods.

The first paper is concerned with the relationship of exact and stable ∞-categories. Any stable ∞-category has a maximal exact structure, but not every exact ∞-category is stable. In this paper we functorially construct a universal *presentable stable envelope* of a small exact ∞-category and study the properties of this association. In particular, we show that it is symmetric monoidal. Working from this starting point, we focus on algebraic K-theory of exact ∞-categories and prove fundamental properties, which were already known for the analogous theory for stable ∞-categories. Finally, we prove that the stable envelope induces an equivalence on algebraic K-theory, effectively translating the study of algebraic K-theory of exact ∞-categories into the study of algebraic K-theory of stable ∞-categories. This is joint work with Christoph Winges.

In the second paper we shift our focus to tensor-triangulated ge-

ometry, or tt-geometry for short, where one assumes that the stable ∞-category in question is equipped with a compatible symmetric monoidal structure and satisfies certain rigidity properties. In tt-geometry, following work of Balmer, one studies a topological space called the Balmer spectrum, which is a generalization of the Zariski spectrum of prime ideals from algebraic geometry. However, one is often interested in studying certain completions of tensor-triangulated categories where the aforementioned rigidity properties need not be satisfied. This is analogous to the theory of formal schemes, a generalization of schemes in which one takes completeness into account. In this paper, we introduce a *formal* version of the Balmer spectrum, and study its fundamental properties. Finally, we prove formal versions of the Hopkins-Neeman theorem and Thomason's theorem, providing comparison theorems between the formal spectrum of the completed derived category of a formal scheme and the formal scheme itself, as a locally ringed space. This is joint work with Drew Heard.

The third and final paper studies interactions between spectral algebraic geometry and the theory of "synthetic spectra" introduced by Pstrągowski. The category of synthetic spectra is in some sense a homotopical categorification of Adams spectral sequence and the theory has led to numerous advances in the computation of stable homotopy groups of the sphere and in the theoretical study of manifolds and multiplicative structures on Moore spectra. One perspective on the Adams spectral sequence is that it is the "descent spectral sequence" of a certain stack. This stack is in some sense a moduli stack classifying the descent data used to define the Adams-spectral sequence in question. In the case of the Adams–Novikov spectral sequence the stack is a spectral refinement of the moduli stack of formal groups. Inspired by this perspective, this paper introduces the notion of a filtered stack and its associated theory of quasi-coherent sheaves. This theory provides a refinement of spectral algebraic geometry, where one keeps track of the descent spectral sequence in addition to coherent cohomology. Included in this paper is a proof of faithfully flat descent for filtered stacks, and as a consequence, we deduce descent for the π_*-flat topology on $\mathbb{E}_\infty$-ring spectra. The final part of the paper focuses on two different applications. First we introduce a certain filtered stack, which encodes the Adams spectral sequence and we produce a comparison functor from synthetic spectra to the category of quasi-coherent sheaves on this filtered stack. This functor is not an equivalence in general, but it is fully faithful on a reasonable subcategory of synthetic spectra. We then focus on the "even filtration" of Hahn–Raksit–Wilson and show that it is encoded by a filtered stack. Building on this, we construct a version of the even filtration for any spectral stack.

Chapter 2

Sammendrag

Set over en bred kam fokuserer denne afhandling på at studere stabile ∞-kategorier via geometriske metoder. Mere specifikt kommer dette til udtryk gennem studiet af algebraisk K-teori, Balmer spektret af en tensor-trianguleret kategori og spektral algebraisk geometri. Dette har resulteret i tre artikler, der hver især fokuserer på en af disse metoder.

I den første artikel er fokus på forholdet mellem eksakte og stabile ∞-kategorier. Enhver stabil ∞-kategori har en maksimal eksakt struktur, men ikke alle eksakte ∞-kategorier er stabile. I denne artikel konstruerer vi funktorielt et *universelt præsentabelt stabilt hylster* fra en lille eksakt ∞-kategori og studerer egenskaber ved denne association. Herunder viser vi, at denne konstruktion kan gøres symmetrisk monoidal. Derefter skifter vi fokus til at studerer algebraisk K-teori for eksakte ∞-kategorier og beviser grundlæggende resultater. Disse var allerede kendt i den analoge teori for stabile ∞-kategorier. Til slut viser vi at det stabile hylster inducerer en ækvivalens på algebraisk K-teori, som effektivt oversætter studiet af algebraisk K-teori for eksakte ∞-kategorier til studiet af algebraisk K-teori for stabile ∞-kategorier. Denne artikel er skrevet i samarbejde med *Christoph Winges*.

I den anden artikel skifter vi fokus til tensor-trianguleret geometri.

Her antager man også at ens stabile ∞-kategori er udstyret med en kompatibel symmetrisk monoidal struktur og at den tilfredsstiller særlige rigiditetsegenskaber. I tt-geometri studerer man et topologisk rum ved navn Balmer spektrum, navngivet efter Balmer som introducerede det i sit skældsættende arbejde. Balmer spektret er en generalisering af Zariski spektret bestående af primidealler som man kender fra algebraisk geometri. Desværre er man ofte interesseret i at studere bestemte fuldstændiggørelser af tt-kategorier, men disse bibeholder sjældent de rigiditets egenskaber vi havde til at starte med. Dette er analogt med teorien af formelle skemaer, en generalisering af skemateori, hvor man også tager fuldstændiggørelse med i sine betragtninger. I denne artikel introducerer vi en *formel* udgave af Balmer spektret, og studerer dettes fundamentale egenskaber. Til slut viser vi formelle versioner af Hopkins-Neeman-sætningen og Thomason-sætningen, hvilket giver sammenligningsresultater af det formelle spektrum af den fuldstændiggjorte deriverede kategori af et formelt skema og det formelle skema selv, som lokalringede rum. Denne artikel er skrevet i samarbejde med *Drew Heard*.

Den tredje og sidste artikel studerer interaktioner mellem spektral algebraisk geometri og teorien om "syntetiske spektra" introduceret af Pstrągowski. Kategorien af syntetiske spektra er på en vis måde en homotopisk kategorificering af Adams-spektralfølgen og teorien har ført til adskillige fremskridt i udregninger af stabile homotopigrupper af sfæren og i det teoretiske studie af mangfoldigheder og multiplikative strukturer på Moore spektra. Et perspektiv på Adams spektral følgen er at den er "nedstigningsspektralfølgen" kommende fra en bestemt stack. Denne stack kan ses som en modulistack der kontrollerer det "nedstignings data" som skal til for at definere Adams-spektralfølgen i fokus. I tilfældet af Adams–Novikov-spektralfølgen er denne stack en spektral udgave af modulistacken af formelle grupper. Inspireret af dette perspektiv, introducerer vi i denne artikel definitionen af en filtreret stack og dennes inducerede teori af kvasi-koherente knipper.

Denne teori kan ses som en refineret udgave af spektral algebraisk geometri, hvor man holder styr på nedstigningsspektralfølgen og ikke kun den koherente kohomologi. Denne artikel indeholder blandt andet et bevis for nedstigning med hensyn til den flade topologi på filtrerede stacks, og som en konsekvens af dette beviser vi nedstigning for π_*-flade topologi på $\mathbb{E}_\infty$-ringe. Den sidste del af denne artikel fokuserer på to forskellige anvendelser af denne teori. Først introducerer vi en bestemt filtreret stack, som kontrollerer Adams-spektralfølgen og vi konstruerer en funktor som sammenligner kategorien af syntetiske spektra med kategorien af kvasi-koherente knipper på denne filtrerede stack. Denne funktor er ikke generelt en ækvivalens af kategorier, men den er fuld og trofast på en rimelig delkategori af syntetisk spektra. Derefter fokuserer vi på den "lige filtrering" introduceret af Hahn–Raksit–Wilson og viser at den er kontrolleret af en filtreret stack. På baggrund af dette definerer vi en version af den lige filtrering på alle spektrale stacks.

Chapter 3

Structural information

This thesis consists of three papers:

1. *The presentable stable envelope of an exact category.* This paper is joint work with *Christoph Winges* and is submitted for publication.

2. *The formal spectrum of a tensor-triangulated category.* This paper is joint work with my supervisor *Drew Heard* and is submitted for publication.

3. *On filtered stacks and the geometrization of synthetic spectra.*

Before the papers, a section providing some of the background needed for understanding the main results of the thesis is included. In addition, the background section includes some informal statements about the main results of this thesis.

Chapter 4

Background and results

The language of this thesis is that of ∞-categories. The definition of an ∞-category is complicated and not informative for this section. For this reason, as is popular in modern mathematics, we will take model-agnostic approach to ∞-categories in this thesis and rather consider ∞-categories and their theory as an foundational building block of mathematics. For unfamiliar readers we suggest the following mental image: ∞-category theory is a homotopical refinement of category theory, where the set of morphisms between two objects is replaced by a homotopy type and compositions of arrows are homotopically coherent.

Because of the above reasoning, this section focuses on the theory of stable ∞-categories and "higher algebra".

Definition 4.0.1. An ∞-category $\mathcal{C}$ is *pointed* if there exists an object $0 \in \mathcal{C}$ which is both initial and terminal. Furthermore, a pointed ∞-category $\mathcal{C}$ is stable if it admits finite limits and colimits and for every pair of composable arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that $gf \simeq 0$ the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\
0 \arrow[r] & Z
\end{tikzcd}$$

is cartesian if and only if it is cocartesian. Such a triple is called an *exact sequence* in $\mathcal{C}$. A functor of stable ∞-categories $F\colon \mathcal{C} \to \mathcal{D}$ is *exact* if it preserves the 0 object and sends exact sequences to exact sequences. This is equivalent to preserving finite limits or colimits.

Remark 4.0.2. Stable ∞-categories are analogous to abelian categories, where the kernel-cokernel pair condition has been replaced with the condition above. To any category (with finite limits) one can associate an abelian category by taking abelian group objects. In the case of the category of sets, the resulting category is the category of abelian groups. Such a procedure exists for ∞-categories too. In this case, the role of the category of sets is played by the ∞-category of spaces, $\mathcal{S}$, (or homotopy types/∞-groupoids/anima if you prefer) and the resulting ∞-category is the ∞-category of *spectra*, usually denoted $\mathcal{S}p$. We believe the analogy shown in

category theory	∞-category theory
Sets	Spaces
Abelian groups	Spectra

is excellent to keep in mind for the rest of this section. Readers familiar with the material will recall that there is also a strong analogy between spectra and cohomology theories. This analogy will be relevant when discussing synthetic spectra at the end of this section.

Example 4.0.3. Every abelian group A determines a spectrum HA, determined by the cohomology theory given by singular cohomology with coefficients in A. The association $A \mapsto HA$ defines a functor

$$H(-)\colon \mathrm{Ab} \to \mathcal{S}p$$

of ∞-categories. This functor is usually called the Eilenberg–Mac Lane functor and we say that the spectrum HA is the Eilenberg–Mac Lane spectrum of A.

Remark 4.0.4. The category of abelian groups can be equipped with a symmetric monoidal structure given by the tensor product of abelian groups determining a functor

$$\mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$$

sending a pair (A, B) to $A \otimes_{\mathbb{Z}} B$. The tensor product has a "unit" namely, the integers $\mathbb{Z}$ considered as a free abelian group with one generator. A simple but effective observation is that the building blocks of modern abstract algebra, can be expressed in terms of algebra objects for the tensor product of abelian groups and modules over these algebra objects with respect to this symmetric monoidal structure. In particular, this allows for the study of algebra in any such symmetric monoidal (∞-)category, where we have a well behaved "tensor product".

In a similar fashion, the ∞-category of spectra can be equipped with a symmetric monoidal structure given by the tensor product of spectra[1]. We denote the tensor product of spectra by $A \otimes B$. The tensor unit is given by the sphere spectrum $\mathbb{S}$. A key insight of Waldhausen is that we should pursue the study of algebra, with respect to the tensor product of spectra. He dubbed this *brave new algebra*, and it often goes under the epithet *higher algebra* in modern mathematics.

The Eilenberg–Mac lane functor is lax symmetric monoidal and in particular, every (commutative) ring is sent to a (commutative) algebra object in the ∞-category of spectra. These are also often called $\mathbb{E}_1$-ring spectra or $\mathbb{E}_\infty$-ring spectra in the commutative case. A theorem of Schwede–Schipley shows that for a (commutative) ring R, the ∞-category of HR-module spectra is equivalent to the derived

[1] This is also often called the *smash product* of spectra.

∞-category of R, usually denoted $D(R)$. This result means that the study of homological algebra of R-modules coincides with the study of the stable homotopy theory of HR-module spectra. However, the unit map $\mathbb{S} \to H\mathbb{Z}$ is not an equivalence and as such, there is a difference between (derived) algebra and "higher algebra". This observation has led to many interesting advances in stable homotopy theory and the rest of mathematics.

4.1 The presentable stable envelope of an exact category

While many stable ∞-categories are related to $\mathcal{S}p$ via "algebra", there are even more that are not. One tool for studying such categories is algebraic K-theory. The first paper of this thesis studies exact ∞-categories, a "type" of ∞-category closely related to stable ∞-categories and their algebraic K-theory.

Definition 4.1.1. If $\mathcal{C}$ is a stable ∞-category, then a full subcategory $\mathcal{E}$ is *extension closed* if it contains the 0 object and for every exact sequence $X \to Y \to Z$ such that $X, Z \in \mathcal{E}$, then we have that $Y \in \mathcal{E}$.

Definition 4.1.2. An ∞-category $\mathcal{C}$ is *additive* if it is pointed, has finite products and coproducts which agree and the homotopy category $h\mathcal{C}$ is an additive 1-category.

Definition 4.1.3. An *exact* ∞-category consists of a triple $(\mathcal{C}, \mathcal{C}_{\mathrm{cof}}, \mathcal{C}_{\mathrm{fib}})$, given by an additive ∞-category $\mathcal{C}$ and two wide subcategories $\mathcal{C}_{\mathrm{cof}}$ and $\mathcal{C}_{\mathrm{fib}}$ whose morphisms are called *cofibrations* and *fibrations* respectively. Furthermore, for every object X the map $0 \to X$ should be a cofibration and $X \to 0$ should be a fibration. Furthermore, we require that pushouts of cofibrations exist and are cofibrations. Likewise, we require that pullbacks of fibrations exist and are fibrations. Finally, we assume that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps, such that $gf \simeq 0$, f is a cofibration and g is a fibration, then the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

is cocartesian if and only if it is cartesian. In this case we say that $X \to Y \to Z$ is an *exact sequence*[1]. An additive functor $F \colon \mathcal{C} \to \mathcal{D}$ of exact ∞-categories is *exact* if it preserves exact sequences.

Remark 4.1.4. Heuristically, we imagine an exact ∞-category as a stable ∞-category, where it is no longer the case that every morphism admits a fiber and cofiber. However, it still comes equipped with a class of exact sequences, and one can show that the data of an exact structure on an additive ∞-category is equivalent to choosing a collection of exact sequence subject to certain conditions. In fact, the *Gabriel–Quillen* embedding theorem for exact ∞-categories proven in the first paper of this thesis, shows that any exact ∞-category is equivalent to an extension closed subcategory of a stable ∞-category and there is an initial presentable stable ∞-category with this property.

Remark 4.1.5. Exact categories were introduced by Quillen to study algebraic K-theory. Likewise exact ∞-categories were introduced by Barwick, to study the algebraic K-theory and prove the theorem of the heart. Since then the main focus in the study of algebraic K-theory has been algebraic K-theory of stable ∞-categories. The study of

[1] This definition is slightly stronger than that of an exact sequence in a stable ∞-category, since not every bifiber sequence in an exact ∞-category is exact in terms of the above definition. Only the bifiber sequences where the first map is a cofibration and the second map is a fibration are exact with respect to the "exact structure" that we are considering.

algebraic K-theory is a vast subject, and we will not attempt to do it justice in this section. For stable ∞-categories, algebraic K-theory defines a functor

$$K\colon \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{S}p$$

for the ∞-category of small stable ∞-categories with exact functors to the ∞-category of spectra.

Work of Blumberg–Gepner–Tabuada shows that the algebraic K-theory functor above is governed by a universal property. Namely, it is the initial *additive invariant*. There exists a refinement called non-connective algebraic K-theory, usually denoted $\mathbb{K}(-)$, which is similarly governed by a universal property, namely it is the initial *localizing invariant*. Near the end of the first paper, we define a non-connective variant of algebraic K-theory of exact ∞-categories and show that it too is governed by a similar universal property. Finally, we prove a higher categorical analogue of the *Gillet–Waldhausen* theorem, showing that the map $\mathcal{C} \to \mathrm{St}(\mathcal{C})$ from an exact ∞-category to its stable envelope induces an equivalence on non-connective algebraic K-theory. That is, the map of spectra

$$\mathbb{K}(\mathcal{C}) \to \mathbb{K}(\mathrm{St}(\mathcal{C}))$$

is an equivalence.

4.2 The formal Balmer spectrum of a tensor-triangulated category

A different tool for understanding stable ∞-categories is the Balmer spectrum and its tensor-triangulated geometry. For this subsection we assume that $\mathcal{C}$ is a small stably symmetric monoidal ∞-category subject to certain rigidity conditions. That is $\mathcal{C}$ is equipped with a tensor product satisfying similar properties to the tensor product of spectra or the derived tensor product of chain complexes.

Definition 4.2.1. A full subcategory $\mathcal{D} \subseteq \mathcal{C}$ is *thick* if it is a stable subcategory and closed under retracts in $\mathcal{C}$. Furthermore, we say that $\mathcal{D}$ is a $\otimes$-*ideal* if for all $X \in \mathcal{D}$ and $Y \in \mathcal{C}$ the tensor product $X \otimes Y$ is in $\mathcal{D}$. Furthermore, we say that a proper $\otimes$-ideal $\mathcal{D}$ is a *prime* $\otimes$-ideal if $X \otimes Y \in \mathcal{D}$ implies that $X \in \mathcal{D}$ or $Y \in \mathcal{D}$. The Balmer spectrum of $\mathcal{C}$, denoted $\operatorname{Spc} \mathcal{C}$, is the set of prime $\otimes$-ideals in $\mathcal{C}$.

Remark 4.2.2. The Balmer spectrum of $\mathcal{C}$ can be given a topology by defining the support of objects to be a basis for the closed subsets and it can even be given a sheaf of commutative rings. If $\mathcal{C} = \operatorname{Perf}(R)$ is the subcategory of the derived ∞-category of a commutative ring R spanned by perfect complexes, then the Hopkins–Neeman theorem proves that there is an isomorphism of locally ringed spaces

$$\operatorname{Spc} \operatorname{Perf}(R) \xrightarrow{\sim} \operatorname{Spec}(R)$$

where $\operatorname{Spec}(R)$ denotes the Zariski spectrum of prime ideals in R with its structure sheaf. This was extended by Thomason to include quasi-compact quasi-separated schemes.

Remark 4.2.3. If $\mathcal{C} = \mathcal{K}^\omega$ for some *large* tt-category $\mathcal{K}$ and $X \in \mathcal{C}$, then one is often interested in studying the Balmer spectrum of the Bousfield localization of $\mathcal{K}$ at X. Unfortunately, this localization can fail to preserve the rigidity conditions we assumed earlier. For example consider the derived ∞-category of the integers $D(\mathbb{Z})$ and the Bousfield localization at $\mathbb{Z}/p$. This is the full subcategory spanned by *derived* p-complete objects. One would hope that this agrees with the derived ∞-category of the p-adic integers $\mathbb{Z}_p$. Unfortunately, these ∞-categories are not equivalent. One has $\mathbb{Z}/p$ as a compact generator, while the other has $\mathbb{Z}_p$ as a compact generator. Only after localizing $D(\mathbb{Z}_p)$ at $\mathbb{Z}/p$ do the ∞-categories become equivalent.

This is analogous to what happens in algebraic geometry, where we would want the spectrum of $\mathbb{Z}_p$ to be a “nilpotent thickening” of the spectrum of $\mathbb{F}_p$. This can be seen to not be the case, since $\operatorname{Spec} \mathbb{Z}_p$ has two points and $\operatorname{Spec} \mathbb{F}_p$ only has one. The solution in algebraic geometry is to consider the *formal spectrum* of $\mathbb{Z}_p$, where one takes the p-adic topology on $\mathbb{Z}_p$ into account. We will discuss this now.

Definition 4.2.4. Suppose R is a topological ring, where the topology of R has a finitely generated ideal of definition $\mathcal{J}$. The *formal spectrum* of R, denoted $\operatorname{Spf} R$ is the subspace of $\operatorname{Spec} R$ spanned by those prime ideals that are open in the topology on R. This can be given a structure sheaf by $\mathcal{J}$-completing the structure sheaf on $\operatorname{Spec} R$ pointwise.

Example 4.2.5. The formal spectrum of $\mathbb{Z}_p$ has a single point given by (p), and the structure sheaf is given by $\mathbb{Z}_p$. Moreover, for any R and finitely generated ideal $\mathcal{J}$, there is an isomorphism of formal affine schemes

$$\operatorname{Spf} R^{\wedge}_{\mathcal{J}} \to \operatorname{Spf} R$$

induced by $\mathcal{J}$-adic completion.

In the second paper, we define the formal spectrum of a pair $(\mathcal{C}, I)$, consisting of a tt-category $\mathcal{C}$ and a choice of Thomason subset I of $\operatorname{Spc} \mathcal{C}$. One should consider the data of the Thomason subset I as a replacement for the $\mathscr{J}$-adic topology on R, in the example above. We then show that the formal spectrum is invariant under completion, and afterwards we prove formal versions of the Hopkins–Neeman theorem and Thomason’s theorem. We also compute the formal spectrum in simple examples coming from representation theory and stable homotopy theory.

4.3 On filtered stacks and the geometrization of synthetic spectra

In the final paper, we center our focus on the ∞-category of spectra and spectral algebraic geometry. Spectral algebraic geometry is a version of algebraic geometry where the role of commutative rings as basic building blocks is replaced by $\mathbb{E}_\infty$-ring spectra. This extends classical algebraic geometry via the Eilenberg–Mac Lane functor, but contains many more objects! We are not able to give a proper introduction in this thesis, but we include some relevant background information and attempt to provide some perspectives on other things relevant to the third paper.

Remark 4.3.1. From any spectrum X one can construct a collection of abelian groups $\{\pi_*(X)\}_{*\in\mathbb{Z}}$, called the *homotopy groups* of X. If X is an HR-module spectrum for some ring R, then the homotopy groups of X are given by the homology groups of the associated object in the derived category of R. If $X = \mathbb{S}$ the sphere spectrum, then the homotopy groups are the *stable homotopy groups of the sphere.*

A major study of stable homotopy theory is the problem of understanding the stable homotopy groups of the sphere. The most successful approach to this problem is using the Adams spectral sequence: A spectral sequence whose input is given by the coherent cohomology of a certain stack M_{fg}, coming from algebra, and whose output computes the stable homotopy groups of the sphere spectrum. In recent years, this method has been improved upon by the introduction of the "cofiber of τ"-methods and the theory of synthetic spectra. The ∞-category of synthetic spectra, usually denoted Syn, is a "categorical deformation" of the Adams spectral sequence, which allows for homotopy theoretic methods to be applied in the study of the Adams spectral sequence. A simpler example of such a categorical deformation is the ∞-category of filtered spectra $\mathcal{S}p^{\mathrm{fil}} := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{S}p)$. This ∞-category is a homotopical refinement of the category of spectral sequences, and will serve as the base for most of the considerations in the third paper.

In the above, we made an analogy between abelian groups and spectra, but spectra were originally defined to as a homotopical refinement of cohomology theories. While these ideas might seem like they are at odds with one another, we consider this reconciled by the relationship between Eilenberg–Mac Lane spectra and singular cohomology. We encourage the reader to keep the following analogy in mind:

Algebra	Homotopy theory
Cohomology theories	Spectra
Spectral sequences	Filtered spectra
Adams spectral sequences	Synthetic spectra

Like the ∞-category of spectra, the ∞-category of synthetic spectra is truly homotopical, but in some sense it is also much more closely related to the homological algebra necessary for computing the cohomology groups that make up the E_2-page of the Adams spectral sequence. This is part of what has made this theory very successful both in computation and in theoretical applications.

One useful perspective on the Adams spectral sequence is that it is the "descent spectral sequence" for a "spectral" stack M_{MU}. The spectral stack M_{MU} is related to the ordinary stack M_{fg} via Quillen's theorem, which identifies the homotopy groups of MU with the Lazard ring, the spectrum of which is a flat cover of M_{fg}. In the third paper, our goal is to make this perspective into a definition and use it to introduce a category of "synthetic" quasi-coherent sheaves for any spectral stack. This is done by introducing the notion of "filtered stacks" and studying their sheaf theory.

Remark 4.3.2. The ∞-category of filtered spectra admits a symmetric monoidal structure given by Day convolution. We say that a commutative algebra object for this monoidal structure is a *commutative filtered* *ring spectrum.* In the first major theorem of the third paper we show that the (opposite) ∞-category of commutative filtered ring spectra admits a Grothendieck topology, which we call the *flat* topology.

Definition 4.3.3. A *filtered stack* is a functor

$$X\colon \operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}}) \to \mathcal{S},$$

which is a sheaf for the flat topology. We denote the ∞-category of filtered stacks by FStack.

The main source of examples of filtered stacks comes from spectral stacks. More precisely, we prove that there is a colimit preserving functor

$$(-)^{\tau}\colon \mathrm{Stk} \to \mathrm{FStack}$$

from the ∞-category of spectral stacks to the ∞-category of filtered stacks, which takes "affines to affines". We also show that this functor satisfies several excellent properties, for instance it preserves flat morphisms and the property of being "geometric".

We then move on to study the ∞-category of quasi-coherent sheaves on a filtered stack. For a filtered stack X^{τ} arising from spectral stacks via the above construction, $\mathrm{QCoh}(X^{\tau})$ is our candidate for the ∞-category of "synthetic" quasi-coherent sheaves on X. We then prove that this is a categorical deformation of the ∞-category of $\mathrm{QCoh}(X)$, in an appropriate sense. One important result that we would like to highlight is that the coherent cohomology of a filtered stack canonically comes equipped with a filtration and for filtered stacks coming from spectral stacks, this is precisely the filtration encoding the "descent spectral sequence".

We then go on to study the relationship of Syn and $\mathrm{QCoh}(M^{\tau}_{\mathrm{MU}})$, and more generally E-based synthetic spectra, Syn_E and $\mathrm{QCoh}(M^{\tau}_{E})$, for an Adams-type ring spectrum E. We construct a comparison functor

$$\mathrm{Syn}_E \to \mathrm{QCoh}(M^{\tau}_{E}),$$

which we show is fully faithful on the full subcategory of Syn_E, where the E-based Adams spectral sequence is weakly convergent. However, this functor can fail to be an equivalence in general.

The final part of the paper is focused on relating the theory of filtered stacks with the "even filtration" of Hahn–Raksit–Wilson. This filtration is very important in the theory of trace methods for algebraic K-theory, but also has strong relations to the Adams spectral sequence. In particular, we show that for any $\mathbb{E}_\infty$-ring spectrum R, the even filtration of R is the filtered coherent cohomology of a certain filtered stack $\mathrm{Spev}^\tau R$. Furthermore, based on these ideas we define an even filtration associated to any spectral stack via descent.

Chapter 5

Acknowledgments

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Part I

The presentable stable envelope of an exact ∞-category

The presentable stable envelope of an exact ∞-category

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Abstract. We prove an analogue of the Gabriel–Quillen embedding theorem for exact ∞-categories, giving rise to a presentable version of Klemenc's stable envelope of an exact ∞-category. Moreover, we construct a symmetric monoidal structure on the ∞-category of small exact ∞-categories and discuss the multiplicative properties of the Gabriel–Quillen embedding. For E an Adams-type homotopy associative ring spectrum, this allows us to identify the symmetric monoidal ∞-category of E-based synthetic spectra with the presentable stable envelope of the exact ∞-category of compact spectra with finite projective E-homology.

In addition, we show that algebraic K-theory, considered as a functor on exact ∞-categories, admits a unique delooping as a localising invariant.

Contents

Chapter I.1

Introduction

Exact ∞-categories were introduced by Barwick in [Bar15] generalising the notion of exact categories as introduced by Quillen in [Qui73]. Every stable ∞-category admits the structure of an exact ∞-category and Klemenc proves in [Kle22] that the functor

$$\mathrm{Cat}^{\mathrm{st}} \to \mathrm{Exact}$$

sending a small stable ∞-category to its associated exact ∞-category admits a left adjoint. Let us denote this functor by $\mathcal{D}^{\mathrm{b}}(-)$ and name it the *stable envelope*. Klemenc also shows that the unit map

$$\mathcal{C} \to \mathcal{D}^{\mathrm{b}}(\mathcal{C})$$

is fully faithful and exhibits $\mathcal{C}$ as an extension-closed subcategory of its stable envelope.' However, $\mathcal{D}^{b}(\mathcal{C})$ is manifestly a small category.

In the case of a small exact 1-category $\mathcal{E}$, the *Gabriel–Quillen embedding*

$$\mathcal{E} \to \mathcal{P}^{\mathrm{Ab}}_{\mathrm{lex}}(\mathcal{E})$$

exhibits $\mathcal{E}$ as an extension-closed subcategory of a Grothendieck abelian category. Here $\mathcal{P}^{\mathrm{Ab}}_{\mathrm{lex}}(\mathcal{E})$ denotes the category of abelian group-valued left exact presheaves. See [Qui73, Section 2], [TT90b, Appendix A], [Büh10, Appendix A] or [KL15, Section 2] for various treatments of this embedding.

In this paper we establish an analogous version of the Gabriel–Quillen embedding pertaining to exact ∞-categories. More precisely, we prove:

I.1.0.1 Theorem (Gabriel–Quillen embedding theorem, I.2.0.7 and I.2.0.11)**.** Let $\mathcal{C}$ be a small exact ∞-category. Then the Yoneda embedding $\mathcal{C} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ exhibits $\mathcal{C}$ as an extension-closed subcategory of a Grothendieck prestable ∞-category.

Furthermore, for every pointed cocomplete ∞-category $\mathcal{D}$, precomposing with the Yoneda embedding induces an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\text{よ}^*} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}).$$

Here $\mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ denotes the full subcategory spanned by left exact presheaves.

This prompts the following definition.

I.1.0.2 Definition. If $\mathcal{C}$ is an exact ∞-category, then the *presentable stable envelope* of $\mathcal{C}$ is defined as

$$\mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}(\mathcal{C}) := \mathrm{Sp}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C})).$$

The presentable stable envelope is the initial presentable stable ∞-category with an exact functor from $\mathcal{C}$. This generalises Klemenc's construction as we show that $\mathrm{Ind}(\mathcal{D}^{\mathrm{b}}(\mathcal{C})) \simeq \mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}(\mathcal{C})$, see Corollary I.2.0.25.

We then proceed to discuss the multiplicative properties of the Gabriel–Quillen embedding and the stable envelope functor in Chapter I.3, where we prove the following.

I.1.0.3 Theorem (Proposition I.3.0.7)**.** The ∞-category Exact of small exact ∞-categories and exact functors admits a symmetric monoidal structure with unit $\mathrm{Proj}(\mathbb{S})$, the additive ∞-category of finitely generated projective $\mathbb{S}$-modules. Moreover, the functors

$$\mathcal{P}_{\mathrm{lex}} \colon \mathrm{Exact} \to \mathrm{Groth} \quad \text{and} \quad \mathcal{P}^{\mathrm{st}}_{\mathrm{lex}} \colon \mathrm{Exact} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$$

refine to symmetric monoidal functors.

In Chapter I.4, we focus on the ∞-category Syn_E of E-based synthetic spectra for an Adams-type homotopy associative ring spectrum E introduced by Pstrągowski in [Pst23a]. The ∞-category of synthetic spectra acts as a 1-parameter deformation of the ∞-category of spectra, with respect to the E-based Adams spectral sequence. This ∞-category is originally defined as the category of additive sheaves on the site $\mathrm{Sp}_E^{\mathrm{fp}}$ of compact spectra whose E-homology is finitely generated and projective as an E_*-module. We prove that this has an interpretation in terms of the presentable stable envelope of an exact ∞-category.

I.1.0.4 Theorem (Theorem I.4.0.13)**.** The ∞-category $\mathrm{Sp}_E^{\mathrm{fp}}$ of E-finite projective spectra admits an exact structure where a sequence $X \to Y \to Z$ is exact if and only if the associated sequence on E-homology is a short exact sequence. Moreover, there is an equivalence

$$\mathrm{Syn}_E \simeq \mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}(\mathrm{Sp}_E^{\mathrm{fp}})$$

between the ∞-category of E-based synthetic spectra and the presentable stable envelope of this exact ∞-category.

I.1.0.5 Remark. The authors believe that the relationship between "synthetic"-type categories and exact categories is stronger than shown in this paper. This may be further explored in the future.

In Chapter I.5, we define a notion of *localising invariant* for exact ∞-categories and prove that every space-valued localising invariant deloops uniquely to a spectrum-valued localising invariant, generalising a theorem of Cisinski and Khan for localising invariants of stable ∞-categories [CK].

In particular, this allows us to define a non-connective algebraic K-theory functor $\mathbb{K}$ which enjoys the following universal property.

I.1.0.6 Proposition (Corollary I.5.0.9)**.** Non-connective algebraic K-theory is the initial localising, spectrum-valued invariant under the (suspension spectrum of the) groupoid core.

We also show that this non-connective algebraic K-theory functor coincides with the functors defined previously by Schlichting for exact 1-categories [Sch04] and by Blumberg, Gepner and Tabuada for stable ∞-categories [BGT13a]. Finally, we prove a version of the Gillet–Waldhausen theorem for non-connective algebraic K-theory of exact ∞-categories.

I.1.0.7 Proposition (Gillet–Waldhausen, Corollary I.5.0.10)**.** The unit transformation $\mathcal{C} \to \mathcal{D}^{\mathrm{b}}(\mathcal{C})$ induces an equivalence

$$\mathbb{K}(\mathcal{C}) \xrightarrow{\sim} \mathbb{K}(\mathcal{D}^{\mathrm{b}}(\mathcal{C}))$$

for every exact ∞-category $\mathcal{C}$.

This also implies that $\mathbb{K}$ refines to a lax symmetric monoidal functor, see Corollary I.5.0.11.

Conventions

1. From this point onwards, the word “category” means “∞-category”.

2. We denote by An the category of anima/spaces/∞-groupoids/weak homotopy types.

3. For a small category $\mathcal{C}$, we denote by $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ the category of presheaves on $\mathcal{C}$ and let $よ : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ be the Yoneda embedding.

4. We use the notions of *Waldhausen categories* and *exact categories* as defined in [Bar15]. In particular, we refer to the distinguished morphisms in a Waldhausen category $\mathcal{C}$ as *ingressive* morphisms, and denote this subcategory by $\mathcal{C}_{\mathrm{in}}$. If $\mathcal{C}$ is exact, we refer to the cofibres of ingressive morphisms as *egressive* morphisms, and denote the subcategory of these by $\mathcal{C}_{\mathrm{eg}}$.

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Chapter I.2

The presentable Gabriel–Quillen embedding

I.2.0.1 Definition. Let $\mathcal{C}$ be a small category with finite coproducts.

1. A functor $X\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{X}$ is *additive* if it preserves finite products.

2. Suppose that $\mathcal{C}_{\mathrm{in}}$ is a Waldhausen structure on $\mathcal{C}$. An additive functor $X\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{X}$ is *left exact* if it sends exact sequences to fibre sequences.

Whenever they are defined, denote by $\mathcal{P}_\Sigma(\mathcal{C})$ and $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ the full subcategories of $\mathcal{P}(\mathcal{C})$ spanned by the additive and left exact presheaves, respectively.

The category of additive presheaves $\mathcal{P}_\Sigma(\mathcal{C})$ on a small category with finite coproducts is an accessible localisation of $\mathcal{P}(\mathcal{C})$ [Lur09a, Proposition 5.5.8.10 (1)]. The Yoneda embedding factors over $\mathcal{P}_\Sigma(\mathcal{C})$, and the induced functor $よ_\Sigma\colon \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C})$ preserves finite coproducts [Lur09a, Proposition 5.5.8.10 (2)]. The inclusion functor $\mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$ preserves sifted colimits [Lur09a, Proposition 5.5.8.10 (4)], and the category $\mathcal{P}_\Sigma(\mathcal{C})$ is projectively generated and presentable [Lur09a, Proposition 5.5.8.25].

I.2.0.2 Definition. Let $i\colon x \rightarrowtail y$ be an ingressive morphism in a Waldhausen category $\mathcal{C}$. Then i gives rise to a comparison map

$$\kappa(i)\colon \operatorname{cofib}(\text{よ}_\Sigma(i)) \to \text{よ}_\Sigma(\operatorname{cofib}(i))$$

in $\mathcal{P}_\Sigma(\mathcal{C})$. Define the *elementary acyclic* associated to i as

$$E(i) := \operatorname{cofib}(\kappa(i)).$$

I.2.0.3 Remark. An additive presheaf on a Waldhausen category $\mathcal{C}$ is left exact if and only if it is local with respect to the collection of morphisms

$$\{\kappa(i) \mid i \in \mathcal{C}_{\mathrm{in}}\}.$$

I.2.0.4 Proposition. Let $\mathcal{C}$ be a small Waldhausen category. Then the inclusion $\mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \subseteq \mathcal{P}_\Sigma(\mathcal{C})$ admits an accessible left adjoint $\Lambda\colon \mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ which preserves compact objects. In particular, $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ is a compactly generated presentable category.

The Yoneda embedding of $\mathcal{C}$ factors through $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$, and the resulting functor $\text{よ}_{\mathrm{lex}}\colon \mathcal{C} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ preserves finite coproducts and sends exact sequences to cofibre sequences.

Proof. Since $\mathcal{C}$ is small, Remark I.2.0.3 and [Lur09a, Prop. 5.5.4.15] imply that the inclusion functor $\mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \to \mathcal{P}_\Sigma(\mathcal{C})$ admits an accessible left adjoint. The localisation functor Λ preserves compact objects because the inclusion functor $\mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \subseteq \mathcal{P}_\Sigma(\mathcal{C})$ preserves filtered colimits. By inspection, the Yoneda embedding induces a fully faithful functor $\text{よ}_{\mathrm{lex}}\colon \mathcal{C} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ which preserves finite coproducts and sends exact sequences to cofibre sequences. $\square$

It appears that little more can be said about よ_{lex} for an arbitrary Waldhausen category $\mathcal{C}$. However, the situation improves greatly when $\mathcal{C}$ is exact, as we are going to discuss in the remainder of this section.

The category $\mathcal{P}_\Sigma(\mathcal{C})$ is prestable if and only if $\mathcal{C}$ is additive [Lurb, Proposition C.1.5.7]. In this case, $\mathcal{P}_\Sigma(\mathcal{C})$ is a *Grothendieck prestable category* in the sense that it is prestable and presentable, and filtered colimits commute with finite limits in $\mathcal{P}_\Sigma(\mathcal{C})$ [Lurb, Definition C.1.4.2]. Either as a consequence of the proof of [Lurb, Proposition C.1.5.7] or by applying [GGN15, Corollary 2.10], the forgetful functor $\Omega^\infty\colon \mathrm{Sp}_{\geq 0} \to \mathrm{An}$ then induces an equivalence

$$\mathrm{Fun}^{\oplus}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0}) \xrightarrow{\simeq} \mathcal{P}_\Sigma(\mathcal{C}),$$

where the domain is the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$ spanned by those presheaves which preserve finite products. Note that both limits and colimits in $\mathrm{Fun}^{\oplus}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$ are computed pointwise. In the sequel, we will tacitly identify $\mathcal{P}_\Sigma(\mathcal{C})$ with the category of additive $\mathrm{Sp}_{\geq 0}$-valued presheaves.

Another way to phrase Proposition I.2.0.4 for an exact category $\mathcal{C}$ is to say that we have a semi-orthogonal decomposition

$$\begin{tikzcd} \ker(\Lambda) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{P}_\Sigma(\mathcal{C}) \arrow[l, shift left, "\mathrm{A}"] \arrow[r, shift left, "\Lambda"] \arrow[r, phantom, "\perp"] & \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \arrow[l, shift left, "i_{\mathcal{C}}"] \end{tikzcd}$$

where

$$\mathrm{A}(X) \simeq \mathrm{fib}(X \to i_{\mathcal{C}}\,\Lambda(X))$$

is the fibre of the unit transformation. Since filtered colimits preserve finite limits in $\mathrm{Sp}_{\geq 0}$ and $i_{\mathcal{C}}$ preserves filtered colimits, the same is true for A. In the following, we will typically suppress explicit mention of the inclusion functors.

I.2.0.5 Definition. Let $\mathcal{C}$ be a category and let S be a subcategory of $\mathcal{C}$.

1. A functor $X \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is *weakly S-effaceable* if for every $x \in \mathcal{C}$ and $\xi \in X(x)$ there exists some morphism $s \colon x' \to x$ in S such that
$$s^*(\xi) = 0 \in X(x').$$

2. A presheaf $X \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}_{\geq 0}$ is *weakly S-effaceable* if $\pi_k X$ is weakly S-effaceable for every $k \in \mathbb{N}$.

This notion allows us to give a concise description of the objects $E(i)$.

I.2.0.6 Lemma. Let $\mathcal{C}$ be an exact category and let $X \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}_{\geq 0}$ be an additive presheaf. The following are equivalent:

1. X is an elementary acyclic associated to some ingressive morphism;

2. there exists an egressive morphism $p \colon y \twoheadrightarrow z$ such that
$$X \simeq \pi_0 \operatorname{cofib}(よ_\Sigma(p)) \simeq \operatorname{coker}(\pi_0 よ_\Sigma(y) \to \pi_0 よ_\Sigma(z));$$

3. there exists a morphism $f \colon y \to z$ in $\mathcal{C}$ such that
$$X \simeq \pi_0 \operatorname{cofib}(よ_\Sigma(f)) \simeq \operatorname{coker}(\pi_0 よ_\Sigma(y) \to \pi_0 よ_\Sigma(z))$$
and X is weakly $\mathcal{C}_{\mathrm{eg}}$-effaceable.

Proof. See [SW26, Proposition 3.7] for a proof. □

I.2.0.7 Theorem. Let $\mathcal{C}$ be a small exact category.

1. The localisation functor $\Lambda \colon \mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ preserves finite limits and $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ is Grothendieck prestable.

2. Every discrete, weakly $\mathcal{C}_{\mathrm{eg}}$-effaceable additive presheaf lies in $\ker(\Lambda)$.

3. Every object in $\ker(\Lambda)$ is weakly $\mathcal{C}_{\mathrm{eg}}$-effaceable.

4. The collection $\{E(i)\}_{i \in \mathcal{C}_{\mathrm{in}}}$ of elementary acyclics generates $\ker(\Lambda)$ under colimits and extensions.

5. For every pointed, cocomplete category $\mathcal{D}$, the restriction functor

$$よ^*_{\mathrm{lex}} \colon \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$$

is an equivalence, where $\mathcal{D}$ is equipped with the maximal Waldhausen structure. Moreover, a functor $\mathcal{C} \to \mathcal{D}$ is exact if and only if it preserves finite coproducts and sends exact sequences to cofibre sequences.

Proof. Since $\mathcal{P}_\Sigma(\mathcal{C})$ is Grothendieck prestable, [Lurb, Prop. C.2.3.1] implies that $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ is Grothendieck prestable if Λ preserves finite limits. For assertion (1), it is therefore enough to show that Λ preserves finite limits.

Egressives in $\mathcal{C}$ are closed under pullbacks, so we obtain a quasi-topology on $\mathcal{C}$ in which a covering of an object x is given precisely by an egressive morphism $x' \twoheadrightarrow x$. The corresponding sheafification functor $\mathcal{P}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$ preserves finite limits [Lur09a, Proposition 6.2.2.7 & Corollary 6.2.1.6] and restricts to a left exact localisation functor $\mathcal{P}_\Sigma(\mathcal{C}) \to \mathrm{Sh}_\Sigma(\mathcal{C})$ onto the category of additive sheaves by [Pst23a, Corollary 2.7].

We wish to identify the full subcategory $\mathrm{Sh}_\Sigma(\mathcal{C})$ of additive sheaves with the full subcategory of left exact presheaves. To do so, we use the equivalence $\mathcal{P}_\Sigma(\mathcal{C}) \simeq \mathrm{Fun}^{\oplus}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$. For an exact sequence $x \overset{i}{\rightarrowtail} y \overset{p}{\twoheadrightarrow} z$ in $\mathcal{C}$, we obtain a commutative diagram

$$\begin{tikzcd}
よ_\Sigma(x) \arrow[r, "よ_\Sigma(i)"] \arrow[d, "\mathrm{id}"] & よ_\Sigma(y) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{cofib}(よ_\Sigma(i)) \arrow[d, "\kappa(i)"] \\
よ_\Sigma(x) \arrow[r, "よ_\Sigma(i)"] & よ_\Sigma(y) \arrow[r, "よ_\Sigma(p)"] & よ_\Sigma(z)
\end{tikzcd}$$

The upper row is (pointwise) a cofibre sequence of connective spectra, whereas the lower row is (pointwise) a fibre sequence of connective spectra. It follows that $\kappa(i)$ is 0-coconnected. Since $よ_\Sigma(y) \to \mathrm{cofib}(よ_\Sigma(i))$ is a π_0-surjection, the induced map on π_0 identifies $\pi_0(\mathrm{cofib}(よ_\Sigma(i))$ with the sub-presheaf of $\pi_0 よ_\Sigma(z)$ given by those morphisms which factor over p. Hence $E(i) = \mathrm{cofib}(\kappa(i))$ is discrete, and $\kappa(i)$ itself is precisely the sieve associated to the covering $p\colon y \twoheadrightarrow z$. It follows that an additive presheaf on $\mathcal{C}$ is a sheaf if and only if it is left exact, so the restricted sheafification functor is precisely $\Lambda\colon \mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$. This proves assertion (1). Alternatively, see [Pst23a, Theorem 2.8] for a proof in terms of Čech covers.

Since sheafification involves only limits and filtered colimits, and $\mathrm{Ab} \subseteq \mathrm{Sp}_{\geq 0}$ is closed under both of these operations, the adjunction $\Lambda\colon \mathcal{P}_\Sigma(\mathcal{C}) \rightleftarrows \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) : i_{\mathcal{C}}$ restricts to an adjunction

$$\Lambda^\heartsuit\colon\ \mathrm{Fun}^\oplus(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \rightleftarrows \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) : \mathrm{inc}\,.$$

It is well-known that the kernel of $\Lambda^\heartsuit$ is given precisely by the full subcategory of weakly $\mathcal{C}_{\mathrm{eg}}$-effaceable presheaves [TT90b, Lemma A.7.11]; in fact, this follows directly by unwinding the explicit formula for the sheafification functor. This proves assertion (2).

By [Lurb, Proposition C.2.3.8], the kernel of Λ is a localising subcategory in the sense of [Lurb, Definition C.2.3.3]. If X is now an object in $\ker(\Lambda)$, then $\Lambda^\heartsuit \pi_0 X \simeq 0$ as well, so $\pi_0 X$ is weakly $\mathcal{C}_{\mathrm{eg}}$-effaceable. Since $\ker(\Lambda)$ is a localising subcategory, we conclude from the cofibre sequence $\tau_{\geq 1}X \to X \to \pi_0 X$ that $\tau_{\geq 1}X$ also lies in the kernel of Λ. Since $\tau_{\geq 1}X$ is 1-connective, there exists a cofibre sequence $\Sigma^{-1}\tau_{\geq 1}X \to 0 \to \tau_{\geq 1}X$, and it follows that $\Sigma^{-1}\tau_{\geq 1}X \in \ker(\Lambda)$. Continuing by induction, we conclude that $\pi_k X$ is weakly $\mathcal{C}_{\mathrm{eg}}$-effaceable for every $k \geq 0$, so assertion (3) holds.

We proceed to prove (4). Let $X \in \ker(\Lambda)$. Like every additive presheaf, X admits a filtration

$$X_0 \to X_1 \to \ldots \to X$$

such that $\operatorname{colim}_n X_n \simeq X$ and $\operatorname{cofib}(X_n \to X_{n+1}) \simeq \bigoplus_\alpha \Sigma^n \text{よ}_\Sigma(x_\alpha)$. Since A commutes with filtered colimits, we have $\mathrm{A}(X) \simeq \operatorname{colim}_n \mathrm{A}(X_n)$.

We claim that $\mathrm{A}(X_n)$ is $(n-1)$-truncated. First, $\mathrm{A}(\bigoplus_\alpha \text{よ}_\Sigma(x_\alpha)) \simeq 0$ because A preserves filtered colimits and all representables are left exact. For $n > 0$, we obtain a commutative diagram of additive presheaves

$$\begin{tikzcd}
\mathrm{A}(\bigoplus_\alpha \Sigma^{n-1} \text{よ}_\Sigma(x_\alpha)) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathrm{A}(\bigoplus_\alpha \Sigma^{n} \text{よ}_\Sigma(x_\alpha)) \arrow[d] \\
\bigoplus_\alpha \Sigma^{n-1} \text{よ}_\Sigma(x_\alpha) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \bigoplus_\alpha \Sigma^{n} \text{よ}_\Sigma(x_\alpha) \arrow[d] \\
\Lambda(\bigoplus_\alpha \Sigma^{n-1} \text{よ}_\Sigma(x_\alpha)) \arrow[r] & 0 \arrow[r] & \Lambda(\bigoplus_\alpha \Sigma^{n} \text{よ}_\Sigma(x_\alpha))
\end{tikzcd}$$

in which all rows and columns are fibre sequences due to $\mathcal{P}_\Sigma(\mathcal{C})$ and $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ being prestable. By induction, the top left term is $(n-2)$-truncated, so the top right term is $(n-1)$-truncated.

Since X_0 is a sum of representables, this shows that $\mathrm{A}(X_0)$ is trivial. The chosen filtration on X induces fibre sequences $\mathrm{A}(X_n) \to \mathrm{A}(X_{n+1}) \to \mathrm{A}(\bigoplus_\alpha \Sigma^{n+1} \text{よ}_\Sigma(x_\alpha))$. The right hand term is n-truncated, and the left hand term is $(n-1)$-truncated by induction, so $\mathrm{A}(X_{n+1})$ is n-truncated as claimed.

Thus, it suffices to show that every n-truncated object in $\ker(\Lambda)$ can be obtained from the elementary acyclics through colimits and extensions. This will be shown by an induction on n. For $n = 0$, we have to consider a discrete and weakly $\mathcal{C}_{\mathrm{eg}}$-effaceable presheaf A. Choose a π_0-surjection $f\colon \bigoplus_\alpha \text{よ}_\Sigma(x_\alpha) \to A$. Since A is weakly effaceable, there exist egressives $(p_\alpha)_\alpha$ such that f factors through the cofibre of $\bigoplus_\alpha よ_\Sigma(p_\alpha)$. The induced map further factors through $\bigoplus_\alpha \pi_0 \operatorname{cofib}(よ_\Sigma(p_\alpha))$ by the discreteness of A. Since we have $\pi_0 \operatorname{cofib}(よ_\Sigma(p_\alpha)) \simeq E(i_\alpha)$ for some i_α by Lemma I.2.0.6, this shows that every discrete, weakly effaceable object admits a resolution by objects which are sums of elementary acyclics. In particular, the subcategory of discrete objects in $\ker(\Lambda)$ is generated by the elementary acyclics under colimits.

For an arbitrary n-truncated object $A \in \ker(\Lambda)$, there exists an extension $\tau_{\geq 1}A \to A \to \pi_0(A)$. Since $\tau_{\geq 1}A$ is a suspension of an $(n-1)$-truncated object, it follows by induction that $\tau_{\geq 1}A$ lies in the subcategory generated by the elementary acyclics. We have just seen that the same is true for $\pi_0(A)$, so A itself is also generated by the elementary acyclics under colimits and extensions.

We are left with proving the universal property claimed in (5). By [Lur09a, Proposition 5.5.4.20], restriction along Λ induces an equivalence

$$\Lambda^*\colon \operatorname{Fun}^{\mathrm{L}}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \operatorname{Fun}^{\mathrm{L}}_K(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}),$$

where $K := \{\kappa(i) \mid i \in \mathcal{C}_{\mathrm{in}}\}$ denotes the collection of comparison maps introduced in Definition I.2.0.2.

Since a colimit-preserving functor $f\colon \mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{D}$ inverts every morphism $\kappa(i)$ if and only if $f \circ よ_\Sigma$ sends exact sequences to cofibre sequences, we obtain a pullback square

$$\begin{tikzcd}
\operatorname{Fun}^{\mathrm{colim}}_K(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \arrow[r] \arrow[d] & \operatorname{Fun}^{\sqcup,\mathrm{rex}}(\mathcal{C}, \mathcal{D}) \arrow[d] \\
\operatorname{Fun}^{\mathrm{colim}}(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \arrow[r] & \operatorname{Fun}^{\sqcup}(\mathcal{C}, \mathcal{D})
\end{tikzcd}$$

in which the top right corner denotes the category of functors preserving finite coproducts and sending exact sequences to cofibre sequences.

The bottom horizontal arrow is an equivalence by [Lur09a, Proposition 5.5.8.15], so the top horizontal arrow is also an equivalence. Thus, we obtain an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\sqcup,\mathrm{rex}}(\mathcal{C}, \mathcal{D}).$$

In particular, if $f\colon \mathcal{C} \to \mathcal{D}$ preserves finite coproducts and sends exact sequences to cofibre sequences, it extends to a colimit-preserving functor $F\colon \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \to \mathcal{D}$ with $F \circ \text{よ}_{\mathrm{lex}} \simeq f$. Since よ_{lex} is an exact functor by construction, it follows that f sends all exact squares in $\mathcal{C}$ to pushout squares in $\mathcal{D}$, so f is already an exact functor of Waldhausen categories. This shows that the inclusion $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\sqcup,\mathrm{rex}}(\mathcal{C}, \mathcal{D})$ is an equivalence, finishing the proof. $\square$

I.2.0.8 Remark. In general, $\ker(\Lambda)$ is properly contained in the subcategory of all weakly $\mathcal{C}_{\mathrm{eg}}$-effaceable presheaves. For example, if every morphism in $\mathcal{C}$ is egressive, then every presheaf is weakly $\mathcal{C}_{\mathrm{eg}}$-effaceable since one can restrict along $0 \to x$.

As direct consequences of Theorem I.2.0.7, we can prove a presentable version of the Gabriel–Quillen embedding theorem and recover Klemenc's stable envelope of an exact category.

I.2.0.9 Lemma. Let $\mathcal{C}$ be a small exact category and let

$$X \xrightarrow{f} Y \xrightarrow{q} \text{よ}_{\mathrm{lex}}(z)$$

be a cofibre sequence in $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ with $z \in \mathcal{C}$. Then there exists a commutative diagram

$$\begin{tikzcd} \text{よ}_{\mathrm{lex}}(y) \arrow[d] \arrow[dr, "\text{よ}_{\mathrm{lex}}(p)"] & \\ Y \arrow[r, "q"] & \text{よ}_{\mathrm{lex}}(z) \end{tikzcd}$$

such that $p\colon y \twoheadrightarrow z$ is an egressive morphism in $\mathcal{C}$.

Proof. Denote by C the cofibre of q in $\mathcal{P}_\Sigma(\mathcal{C})$. The pointwise suspension $\Sigma \circ X$ acquires a canonical map $c\colon \Sigma \circ X \to C$ which becomes an equivalence upon application of Λ. Hence $\mathrm{cofib}(c)$ is weakly $\mathcal{C}_{\mathrm{eg}}$-effaceable by Theorem I.2.0.7 (3). In particular, the morphism $よ(z) = よ_{\mathrm{lex}}(z) \to \mathrm{cofib}(c)$ represents an element in $\pi_0(\mathrm{cofib}(c))(z)$, and this element becomes trivial upon precomposition with some egressive $p\colon y \twoheadrightarrow z$. Hence $よ_{\mathrm{lex}}(p)$ lifts to a morphism $よ_{\mathrm{lex}}(y) \to Y$. □

To establish the Gabriel–Quillen embedding theorem, we will also require Quillen's 'obscure axiom'.

I.2.0.10 Lemma. Let $\mathcal{C}$ be an exact category. Suppose that $p\colon x \to y$ is a morphism in $\mathcal{C}$ which admits a fibre and that there exists a morphism $f\colon x' \to x$ such that pf is egressive in $\mathcal{C}$. Then p is egressive.

Proof. See [SW26, Lemma 2.5]. □

I.2.0.11 Theorem (Gabriel–Quillen embedding theorem)**.** Let $\mathcal{C}$ be a small Waldhausen category. The following are equivalent:

1. $\mathcal{C}$ is exact.
2. The category of left exact presheaves $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ is prestable and the Yoneda embedding $よ_{\mathrm{lex}}\colon \mathcal{C} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ identifies $\mathcal{C}$ with an extension-closed subcategory.
3. There exist a prestable category $\mathcal{D}$ and a fully faithful functor $\mathcal{C} \to \mathcal{D}$ which identifies $\mathcal{C}$ with an extension-closed subcategory of $\mathcal{D}$.

Proof. Evidently, (2) implies (3), and (3) implies (1) because extension-closed subcategories of exact categories are exact.

Suppose that (1) holds. Then Theorem I.2.0.7 shows that $よ_{\mathrm{lex}}\colon \mathcal{C} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ is an exact embedding into a Grothendieck prestable category.

We are left with showing that the essential image of $よ_{\mathrm{lex}}$ is closed under extensions and that $よ_{\mathrm{lex}}\colon \mathcal{C} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ reflects exact sequences.

Consider a cofibre sequence $よ_{\mathrm{lex}}(x) \to Y \to よ_{\mathrm{lex}}(z)$ with $x, z \in \mathcal{C}$. By Lemma I.2.0.9, there exists an exact sequence

$$x' \rightarrowtail y \overset{p}{\twoheadrightarrow} z$$

such that p factors over Y. Since $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ is prestable, the resulting commutative square

$$\begin{tikzcd} よ_{\mathrm{lex}}(x') \arrow[r] \arrow[d] & よ_{\mathrm{lex}}(y) \arrow[d] \\ よ_{\mathrm{lex}}(x) \arrow[r] & Y \end{tikzcd}$$

is a pushout, so $Y \simeq よ_{\mathrm{lex}}(x \cup_{x'} y)$ by exactness of $よ_{\mathrm{lex}}$.

Finally, consider a cofibre sequence $よ_{\mathrm{lex}}(x_0) \to よ_{\mathrm{lex}}(x_1) \to よ_{\mathrm{lex}}(x_2)$ in $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$. Applying Lemma I.2.0.9 once more, we find an egressive morphism $y \twoheadrightarrow x_2$ in $\mathcal{C}$ which factors over x_1. Since $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ is prestable, $よ_{\mathrm{lex}}(x_0) \to よ_{\mathrm{lex}}(x_1) \to よ_{\mathrm{lex}}(x_2)$ is a (pointwise) fibre sequence, so $x_1 \to x_2$ admits a fibre. Quillen's obscure axiom Lemma I.2.0.10 implies that $x_1 \to x_2$ is an egressive morphism in $\mathcal{C}$. We conclude that $よ_{\mathrm{lex}}$ reflects exact sequences. $\square$

For the sake of completeness, we also record the following property of the Gabriel–Quillen embedding.

I.2.0.12 Proposition. Let $\mathcal{C}$ be a small exact category. Then $\mathcal{C}$ is closed under fibres of egressives in $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ if and only if $\mathcal{C}$ is weakly idempotent complete.

Proof. The proof given in [SW26, Theorem 1.4] carries over to $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$. $\square$

I.2.0.13 Definition. For any exact category $\mathcal{C}$, define the *presentable stable envelope* of $\mathcal{C}$ to be

$$\mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}(\mathcal{C}) := \mathrm{Sp}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C})).$$

I.2.0.14 Corollary. For every small exact category $\mathcal{C}$ and every presentable, stable category $\mathcal{D}$, the restriction functor

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$$

is an equivalence.

Proof. This is immediate by combining [Lura, Corollary 1.4.4.5] with the universal property of $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ from Theorem I.2.0.7 (5). □

I.2.0.15 Remark. Let $\mathcal{A}$ be a small abelian category. Then $\mathrm{Ind}(\mathcal{A})$ is a Grothendieck abelian category (ie it is abelian, presentable, and filtered colimits are exact), and therefore gives rise to the unbounded derived category $\mathcal{D}(\mathrm{Ind}(\mathcal{A}))$ as defined in [Lura, Definition 1.3.5.8]. By Corollary I.2.0.14 and [Lura, Proposition 1.3.5.9 & 1.3.5.21], the exact functor $\mathcal{A} \to \mathrm{Ind}(\mathcal{A}) \to \mathcal{D}(\mathrm{Ind}(\mathcal{A}))$ induces a colimit-preserving functor

$$\mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}(\mathcal{A}) \to \mathcal{D}(\mathrm{Ind}(\mathcal{A})).$$

Interpreting $\mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}(\mathcal{A})$ as the category of spectrum-valued sheaves on $\mathcal{A}$ with respect to the topology generated by the epimorphisms in $\mathcal{A}$, [Pst23a, Theorem 2.64] identifies this functor as the corresponding hypersheafification functor.

We are not aware of a similar interpretation of the hypersheafification functor in the case of arbitrary exact categories.

The property of よ_{lex} formulated in Lemma I.2.0.9 deserves some attention on its own. Our terminology is borrowed from [Sch04].

I.2.0.16 Definition. Let $\mathcal{C}$ be an exact category. An extension-closed subcategory $\mathcal{U} \subseteq \mathcal{C}$ is *left special* if the following holds: for every egressive morphism $p\colon x \twoheadrightarrow u$ with $x \in \mathcal{C}$ and $u \in \mathcal{U}$, there exists a morphism $f\colon v \to x$ with $v \in \mathcal{U}$ such that the composite $pf\colon v \to u$ is an egressive morphism in $\mathcal{U}$.

A *right special* subcategory is an extension-closed subcategory $\mathcal{U} \subseteq \mathcal{C}$ such that $\mathcal{U}^{\mathrm{op}} \subseteq \mathcal{C}^{\mathrm{op}}$ is left special.

I.2.0.17 Example.

1. Let $\mathcal{C}$ be an additive category with its split-exact structure and let $\mathcal{U}$ be a full additive subcategory. Then $\mathcal{U} \subseteq \mathcal{C}$ is left special and right special.

2. Theorem I.2.0.11 and Lemma I.2.0.9 show that the Yoneda embedding よ$_{\mathrm{lex}}\colon \mathcal{C} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ exhibits $\mathcal{C}$ as a left special subcategory.

3. The full subcategory of torsion groups in the abelian category of finitely generated abelian groups is right special.

A convenient reformulation of Definition I.2.0.16 is the following.

I.2.0.18 Lemma. For $\mathcal{C}$ exact and $\mathcal{U} \subseteq \mathcal{C}$ an extension-closed subcategory, the following are equivalent:

1. $\mathcal{U}$ is left special in $\mathcal{C}$.

2. Every ingressive morphism $i\colon x \rightarrowtail y$ of $\mathcal{C}$ with $\mathrm{cofib}(i) \in \mathcal{U}$ is the pushout of an ingressive morphism in $\mathcal{U}$.

Proof. If $\mathcal{U}$ is left special, we obtain for every ingressive $i\colon x \rightarrowtail y$ a map of exact sequences

$$\begin{tikzcd} u \arrow[r, tail] \arrow[d] & v \arrow[r, two heads] \arrow[d] & \mathrm{cofib}(i) \arrow[d, "\mathrm{id}"] \\ x \arrow[r, tail, "i"] & y \arrow[r, two heads] & \mathrm{cofib}(i) \end{tikzcd}$$

Hence the left square is a pushout. Conversely, the fibre of an egressive $p\colon y \to w$ with $w \in \mathcal{U}$ is an ingressive with cofibre in $\mathcal{U}$, so we obtain an analogous pushout square in which the right square witnesses that $\mathcal{U} \subseteq \mathcal{C}$ is left special. $\square$

For ordinary exact categories, Keller has shown that inclusions of left special subcategories induce fully faithful functors on derived categories [Kel96, Section 12]. On the level of small categories, [SW26, Theorem 1.2] provides an analogous assertion for stable envelopes of exact categories. By considering presentable categories, the meaning of this condition can be described a bit more explicitly. Parts of the following proposition can also be found in [Pst23a, Proposition 2.11 & Corollary 2.44].

I.2.0.19 Proposition. Let $\mathcal{C}$ be a small exact category and let $\mathcal{U} \subseteq \mathcal{C}$ be an extension-closed subcategory. Denote the inclusion functor by $j\colon \mathcal{U} \to \mathcal{C}$. The following are equivalent:

1. $\mathcal{U}$ is left special in $\mathcal{C}$;

2. the restriction functor $j^*\colon \mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{P}_\Sigma(\mathcal{U})$ sends weakly $\mathcal{C}_{\mathrm{eg}}$-effaceable presheaves to weakly $\mathcal{U}_{\mathrm{eg}}$-effaceable presheaves;

3. the square

$$\begin{tikzcd} \mathcal{P}_\Sigma(\mathcal{C}) \arrow[r, "j^*"] & \mathcal{P}_\Sigma(\mathcal{U}) \\ \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \arrow[u, "i_{\mathcal{C}}"] \arrow[r, "j^*"] & \mathcal{P}_{\mathrm{lex}}(\mathcal{U}) \arrow[u, "i_{\mathcal{U}}"'] \end{tikzcd}$$

is (vertically) left adjointable, ie the base change transformation

$$\Lambda_{\mathcal{U}}\, j^* \to j^*\, \Lambda_{\mathcal{C}}$$

is an equivalence;

4. $j^*\colon \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{U})$ preserves colimits.

If these conditions are satisfied, the left adjoint $j_\natural\colon \mathcal{P}_{\mathrm{lex}}(\mathcal{U}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ of j^* is fully faithful.

Proof. To show that (1) implies (2), assume that $\mathcal{U}$ is left special in $\mathcal{C}$ and consider a weakly $\mathcal{C}_{\mathrm{eg}}$-effaceable presheaf $X \in \mathcal{P}_\Sigma(\mathcal{C})$. By definition, there exists for every $u \in \mathcal{U}$ and $\alpha \in \pi_k X(u)$ an egressive morphism $p\colon x \twoheadrightarrow u$ such that $p^*\alpha = 0$. Since $\mathcal{U}$ is left special, there also exists a morphism $f\colon v \to x$ such that the composite pf is an egressive morphism in $\mathcal{U}$, and $(pf)^*\alpha = 0$.

Assume (2). Then j^* clearly sends discrete weakly $\mathcal{C}_{\mathrm{eg}}$-effaceable presheaves to discrete weakly $\mathcal{U}_{\mathrm{eg}}$-effaceable presheaves. Since we know that $j^*\colon \mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{P}_\Sigma(\mathcal{U})$ preserves all colimits, it restricts to the kernels of the sheafification functors by Theorem I.2.0.7 (4). The base change transformation in (3) is given by the composite

$$\Lambda_{\mathcal{U}}\, j^* \to \Lambda_{\mathcal{U}}\, j^* i_{\mathcal{C}}\, \Lambda_{\mathcal{C}} \simeq \Lambda_{\mathcal{U}}\, i_{\mathcal{U}} j^*\, \Lambda_{\mathcal{C}} \xrightarrow{\sim} j^*\, \Lambda_{\mathcal{C}},$$

where the first and last arrow are induced by the respective unit and counit. Therefore, the base change tranformation is an equivalence if the sheafification of the unit morphism $j^*(u)\colon j^* \to j^* i_{\mathcal{C}}\, \Lambda_{\mathcal{C}}$ is an equivalence. This is the case precisely if the cofibre of this morphism lies in $\ker(\Lambda_{\mathcal{U}})$. As $j^*\colon \mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{P}_\Sigma(\mathcal{U})$ preserves cofibre sequences, it follows that $\Lambda_{\mathcal{U}}\, j^*(u)$ is an equivalence, so (3) holds.

To see that (3) implies (4), observe that for every diagram $X\colon I \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ we have

$$\begin{aligned}\operatorname*{colim}_I j^* X &\simeq \Lambda_{\mathcal{U}} \operatorname*{colim}_I i_{\mathcal{U}} j^* X \simeq \Lambda_{\mathcal{U}} \operatorname*{colim}_I j^* i_{\mathcal{C}} X \\ &\simeq \Lambda_{\mathcal{U}}\, j^* \operatorname*{colim}_I i_{\mathcal{C}} X \simeq j^*\, \Lambda_{\mathcal{C}} \operatorname*{colim}_I i_{\mathcal{C}} X \\ &\simeq j^*(\operatorname*{colim}_I X).\end{aligned}$$

Suppose that (4) holds and consider an egressive morphism $p\colon x \twoheadrightarrow u$ with $x \in \mathcal{C}$ and $u \in \mathcal{U}$. Then

$$j^* \text{よ}_{\mathrm{lex}}(\mathrm{fib}(p)) \to j^* \text{よ}_{\mathrm{lex}}(x) \xrightarrow{\text{よ}(p)} j^* \text{よ}_{\mathrm{lex}}(u)$$

is a cofibre sequence in $\mathcal{P}_{\mathrm{lex}}(\mathcal{U})$. Since $j^* \text{よ}_{\mathrm{lex}}(u)$ is representable and $\mathcal{U} \subseteq \mathcal{P}_{\mathrm{lex}}(\mathcal{U})$ is left special by Lemma I.2.0.9, there exists a morphism $f\colon v \to x$ such that the composite pf is egressive in $\mathcal{U}$. So $\mathcal{U}$ is left special in $\mathcal{C}$.

To conclude that $j_\natural$ is fully faithful under these conditions, note that $j_\natural \simeq \Lambda_{\mathcal{C}}\, j_! i_{\mathcal{U}}$, where $j_!\colon \mathcal{P}_\Sigma(\mathcal{U}) \to \mathcal{P}_\Sigma(\mathcal{C})$ is the left adjoint to j^*. The associated unit transformation is given by the composite

$$\mathrm{id} \xrightarrow{\sim} \Lambda_{\mathcal{U}}\, i_{\mathcal{U}} \xrightarrow{\sim} \Lambda_{\mathcal{U}}\, j^* j_! i_{\mathcal{U}} \to j^*\, \Lambda_{\mathcal{C}}\, j_! i_{\mathcal{U}} \simeq j^* j_\natural,$$

the third arrow being the base change transformation. □

I.2.0.20 Corollary. Let $\mathcal{C}$ be a small exact category and let $\mathcal{U} \subseteq \mathcal{C}$ be a left special subcategory. Then there exists a recollement

$$\begin{tikzcd}
\mathcal{P}^{\mathcal{U}}_{\mathrm{lex}}(\mathcal{C}) \arrow[r, hook] & \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \arrow[r, "j^*"] \arrow[l, bend right=60, "\perp"] \arrow[l, bend left=60, "\perp"'] & \mathcal{P}_{\mathrm{lex}}(\mathcal{U}) \arrow[l, bend right=60, "j_\natural"'] \arrow[l, bend left=60, "j_*"]
\end{tikzcd}$$

of Grothendieck prestable categories, where $\mathcal{P}^{\mathcal{U}}_{\mathrm{lex}}(\mathcal{C})$ denotes the full subcategory of left exact presheaves vanishing on $\mathcal{U}$.

Proof. Since $\mathcal{U}$ is left special in $\mathcal{C}$, Proposition I.2.0.19 implies that j^* preserves colimits, so it admits a right adjoint $j_*\colon \mathcal{P}_{\mathrm{lex}}(\mathcal{U}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$. It follows formally that the inclusion of $\mathcal{P}^{\mathcal{U}}_{\mathrm{lex}}(\mathcal{C})$ admits both adjoints, and that these are given by

$$\mathrm{cofib}(j_\natural j^* \Rightarrow \mathrm{id}) \quad \text{and} \quad \mathrm{fib}(\mathrm{id} \Rightarrow j_* j^*),$$

respectively. In particular, $\mathcal{P}^{\mathcal{U}}_{\mathrm{lex}}(\mathcal{C})$ is closed under limits and colimits, so it is Grothendieck prestable. □

It is easy to recover Klemenc's stable envelope of an exact category from the presentable Gabriel–Quillen embedding.

I.2.0.21 Definition. For a small exact category $\mathcal{C}$, denote by $\mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C})$ the smallest full prestable subcategory of $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ containing the essential image of the Yoneda embedding $よ_{\mathrm{lex}}$. Define the *stable envelope*

$$\mathcal{D}^{\mathrm{b}}(\mathcal{C}) := \mathrm{SW}(\mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C}))$$

as the Spanier–Whitehead stabilisation of $\mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C})$.

I.2.0.22 Remark. Proposition I.2.0.19 directly implies [SW26, Theorem 1.2]: the functor $\mathcal{D}^{\mathrm{b}}(\mathcal{U}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{C})$ is fully faithful for every left or right special subcategory $\mathcal{U} \subseteq \mathcal{C}$.

I.2.0.23 Proposition. Let $\mathcal{C}$ be a small exact category and let $\mathcal{D}$ be a prestable category. Restriction along $\mathcal{C} \to \mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C})$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}).$$

In particular, $\mathcal{D}^{\mathrm{b}}_{\geq 0}$ refines to a left adjoint functor to the fully faithful inclusion of small prestable categories to small exact categories.

Proof. For any small exact category $\mathcal{E}$, the image of an exact functor $\mathcal{E} \to \mathcal{D}$ is contained in a small prestable subcategory of $\mathcal{D}$, for example the closure of its image under finite colimits and extensions. Therefore, we have an equivalence

$$\operatorname*{colim}_{\substack{\mathcal{D}' \subseteq \mathcal{D} \\ \text{small prestable subcat.}}} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}, \mathcal{D}') \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}, \mathcal{D})$$

which is natural in $\mathcal{E}$. This allows us to assume without loss of generality that $\mathcal{D}$ is small.

Since $\mathcal{D}^{b}_{\geq 0}(\mathcal{C})$ is generated under finite colimits and extensions by $\mathcal{C}$, the square

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^{b}_{\geq 0}(\mathcal{C}), \mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}) \arrow[d] \\
\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^{b}_{\geq 0}(\mathcal{C}), \mathcal{P}_{\mathrm{lex}}(\mathcal{D})) \arrow[r] & \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{P}_{\mathrm{lex}}(\mathcal{D}))
\end{tikzcd}$$

is a pullback. Since $\mathcal{C} \to \mathcal{D}^{b}_{\geq 0}(\mathcal{C})$ is left special, the induced colimit-preserving functor $\mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{D}^{b}_{\geq 0}(\mathcal{C}))$ is fully faithful, and it is essentially surjective since $\mathcal{D}^{b}_{\geq 0}(\mathcal{C})$ itself is generated by $\mathcal{C}$. It follows from Theorem I.2.0.7 that the lower horizontal arrow in the above diagram is an equivalence. □

I.2.0.24 Corollary. Let $\mathcal{C}$ be a small exact category and let $\mathcal{D}$ be a stable category. Restriction along $\mathcal{C} \to \mathcal{D}^{b}(\mathcal{C})$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^{b}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}).$$

In particular, $\mathcal{D}^{b}$ refines to a left adjoint of the fully faithful inclusion of small stable categories to small exact categories.

Proof. This is immediate from Proposition I.2.0.23 because SW is in particular left adjoint to the inclusion of stable categories into prestable categories. □

I.2.0.25 Corollary. The inclusions $\mathcal{D}^{b}_{\geq 0} \to \mathcal{P}_{\mathrm{lex}}$ and $\mathcal{D}^{b} \to \mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}$ induce equivalences

$$\mathrm{Ind} \circ \mathcal{D}^{b}_{\geq 0} \simeq \mathcal{P}_{\mathrm{lex}} \quad \text{and} \quad \mathrm{Ind} \circ \mathcal{D}^{b} \simeq \mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}.$$

Proof. For every small exact category $\mathcal{C}$ and every Grothendieck prestable category $\mathcal{D}$, we have equivalences

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}(\mathcal{D}^{b}_{\geq 0}(\mathcal{C})), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^{b}_{\geq 0}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$$

by Proposition I.2.0.23. Since $\mathrm{Ind}(\mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C}))$ is Grothendieck prestable, it satisfies the same universal property as $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ by Theorem I.2.0.7 (5).

The second equivalence follows by stabilising. $\square$

Finally, the presentable Gabriel–Quillen embedding also allows for an easy construction of idempotent completions of exact categories as exact categories.

I.2.0.26 Lemma. Let $\mathcal{C}$ be a small exact category.

1. The Gabriel–Quillen embedding $\text{よ}_{\mathrm{lex}}(\mathcal{C})\colon \mathcal{C} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ induces a fully faithful functor $\mathcal{C}^\natural \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ which exhibits the idempotent completion as an extension-closed subcategory.

2. In the resulting exact structure on $\mathcal{C}^\natural$, the exact sequences are precisely the retracts of exact sequences in $\mathcal{C}$.

3. If $\mathcal{U}$ is an extension-closed subcategory of $\mathcal{C}$, then $\mathcal{U}^\natural$ is an extension-closed subcategory of $\mathcal{C}^\natural$.

Proof. The idempotent completion of $\mathcal{C}$ is the retract-closure of $\mathcal{C}$ in $\mathcal{P}(\mathcal{C})$. Since the Yoneda embedding factors over $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ and retracts of left exact presheaves are also left exact, this identifies $\mathcal{C}^\natural$ with a full subcategory of $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$.

Let $\widehat{x} \to Y \to \widehat{z}$ be a cofibre sequence in $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ with $\widehat{x}, \widehat{z} \in \mathcal{C}^\natural$. Then there exist $x', z' \in ({}^\natural\mathcal{C})$ such that $\widehat{x} \oplus x' \simeq x \in \mathcal{C}$ and $\widehat{z} \oplus z' \simeq z \in \mathcal{C}$. From the cofibre sequence $x \to Y \oplus x' \oplus z' \to z$, we conclude that Y is a retract of an object in $\mathcal{C}$ because $\mathcal{C}$ is extension-closed. This argument also proves the characterisation of exact sequences in $\mathcal{C}^\natural$.

If $\widehat{u} \rightarrowtail \widehat{y} \twoheadrightarrow \widehat{w}$ is an exact sequence with $\widehat{u}, \widehat{w} \in \mathcal{U}^\natural$, then stabilising with appropriate objects from $\mathcal{U}^\natural$ as before shows that $\widehat{y} \in \mathcal{U}^\natural$ and exhibits the given sequence as a retract of an exact sequence in $\mathcal{U}$. $\square$

I.2.0.27 Corollary. For every exact category $\mathcal{C}$ and every idempotent complete exact category $\mathcal{D}$, the restriction functor

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^\natural, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$$

is an equivalence. In particular, the inclusion $\mathrm{Exact}^{\mathrm{perf}} \to \mathrm{Exact}$ of the full subcategory of idempotent complete exact categories admits a left adjoint

$$(-)^\natural \colon \mathrm{Exact} \to \mathrm{Exact}^{\mathrm{perf}}.$$

Proof. This is immediate from the universal property of idempotent completion and the characterisation of exact sequences in $\mathcal{C}^\natural$ given in Lemma I.2.0.26. □

To conclude this section, we will show that the formation of stable envelopes preserves idempotent completeness. This can be conveniently shown using the notion of heart structures.

I.2.0.28 Definition (Saunier [Sau, Definition 2.3])**.** A *heart structure* on a stable category $\mathcal{C}$ is a pair of full subcategories $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ satisfying the following properties:

1. the subcategory $\mathcal{C}_{\geq 0}$ is closed under finite colimits and extensions;
2. the subcategory $\mathcal{C}_{\leq 0}$ is closed under finite limits and extensions;
3. for every object $x \in \mathcal{C}$ there exists a cofibre sequence

$$x_{\leq 0} \to x \to x_{\geq 1}$$

   such that $x_{\leq 0} \in \mathcal{C}_{\leq 0}$ and $\Sigma^{-1} x_{\leq 1} \in \mathcal{C}_{\geq 0}$.

For $k \in \mathbb{Z}$, define $\mathcal{C}_{\geq k} := \Sigma^n \mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq k} := \Sigma^k(\mathcal{C}_{\leq 0})$ and $\mathcal{C}_{[k,l]} := \mathcal{C}_{\geq k} \cap \mathcal{C}_{\leq l}$. The *heart* of the heart structure is defined to be

$$\mathcal{C}^\heartsuit := \mathcal{C}_{[0,0]}.$$

The heart structure is *bounded* if

$$\mathcal{C} \simeq \bigcup_{k\in\mathbb{N}} \mathcal{C}_{[-k,k]}.$$

An exact functor $f\colon \mathcal{C} \to \mathcal{D}$ between stable categories with heart structures is *heart-exact* if $f(\mathcal{C}_{\geq 0}) \subseteq \mathcal{D}_{\geq 0}$ and $f(\mathcal{C}_{\leq 0}) \subseteq \mathcal{D}_{\leq 0}$.

Denote by $\mathrm{Cat}^{\heartsuit}$ the category of small stable categories equipped with a bounded heart structure and heart-exact functors between them.

The key observation is the following.

I.2.0.29 Proposition (Sosnilo)**.** Let $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a stable category with a bounded heart structure. Then the following holds:

1. $((\mathcal{C}_{\geq 0})^{\natural}, (\mathcal{C}_{\leq 0})^{\natural})$ defines a bounded heart structure on $\mathcal{C}^{\natural}$;

2. the induced functor $\mathcal{C}^{\heartsuit} \to (\mathcal{C}^{\natural})^{\heartsuit}$ is an idempotent completion.

Proof. The full subcategory $(\mathcal{C}_{\geq 0})^{\natural}$ is closed under finite colimits, and $(\mathcal{C}_{\leq 0})^{\natural}$ is closed under finite limits. If $\mathcal{D} \subseteq \mathcal{C}$ is any extension-closed subcategory, then $\mathcal{D}^{\natural} \subseteq \mathcal{C}^{\natural}$ is also extension-closed by Lemma I.2.0.26 (3).

Let now $x \in \mathcal{C}^{\natural}$ be arbitrary. We claim that there exists a cofibre sequence

$$x_{\leq 0} \to x \to x_{\geq 1}$$

in $\mathcal{C}^{\natural}$ with $x_{\leq 0} \in \mathcal{C}_{\leq 0}$ (without idempotent completion) and $x_{\geq 1} \in (\mathcal{C}_{\geq 1})^{\natural}$.

Choose $y \in \mathcal{C}^{\natural}$ with $x \oplus y \in \mathcal{C}$. Since the heart structure on $\mathcal{C}$ is bounded, there exists some $k \in \mathbb{Z}$ with $x \oplus y \in \mathcal{C}_{\geq k}$. If $k \geq 1$, then $0 \to x \oplus y \xrightarrow{\mathrm{id}} x \oplus y$ is a suitable cofibre sequence, so we only have to consider $k \leq 0$, where we prove the claim by induction.

Choosing a decomposition of $x \oplus y$ in $\mathcal{C}$ induces a bicartesian square

$$\begin{tikzcd}
(x \oplus y)_{\leq 0} \arrow[r] \arrow[d] & x \arrow[d, "f"] \\
y \arrow[r, "g"] & (x \oplus y)_{\geq 1}
\end{tikzcd}$$

Set $x' := \mathrm{cofib}((x \oplus y)_{\leq 0} \to x)$. Since $x \oplus y \in \mathcal{C}_{\geq k}$, we find that

$$x' \oplus \Sigma x \simeq \mathrm{cofib}(g + 0\colon y \oplus x \to (x \oplus y)_{\geq 1}) \in \mathcal{C}_{\geq k+1}.$$

So $x' \in (\mathcal{C}_{\geq k+1})^{\natural}$. By induction, x' admits a decomposition

$$x'_{\leq 0} \to x' \to x'_{\geq 1}$$

with $x'_{\leq 0} \in \mathcal{C}_{\leq 0}$ and $x'_{\geq 1} \in (\mathcal{C}_{\geq 1})^{\natural}$. Consequently, the fibre of the composite map $p\colon x \to x' \to x'_{\geq 1}$ sits in a cofibre sequence $(x \oplus y)_{\leq 0} \to \mathrm{fib}(p) \to x'_{\leq 0}$, which shows that $\mathrm{fib}(p) \in \mathcal{C}_{\leq 0}$. Consequently, $\mathrm{fib}(p) \to x \xrightarrow{p} x'_{\geq 1}$ is the required decomposition.

Note that this argument can be dualised to show that every object $x \in \mathcal{C}^{\natural}$ also admits a decomposition

$$x_{\leq 0} \to x \to x_{\geq 1}$$

in $\mathcal{C}^{\natural}$ with $x_{\leq 0} \in (\mathcal{C}_{\leq 0})^{\natural}$ and $x_{\geq 1} \in \mathcal{C}_{\geq 1}$.

It is immediate that $(\mathcal{C}^{\natural}, (\mathcal{C}_{\geq 0})^{\natural}, (\mathcal{C}_{\leq 0})^{\natural})$ is a bounded heart category.

Note that $\mathcal{C}^{\heartsuit} \to \mathcal{C} \to \mathcal{C}^{\natural}$ induces a fully faithful exact functor $(\mathcal{C}^{\heartsuit})^{\natural} \to \mathcal{C}^{\natural}$, which factors over $(\mathcal{C}^{\natural})^{\heartsuit}$ by definition of the heart structure on $\mathcal{C}^{\natural}$. We have to show that the resulting functor $(\mathcal{C}^{\heartsuit})^{\natural} \to (\mathcal{C}^{\natural})^{\heartsuit}$ is essentially surjective.

Consider first an object $x \in \mathcal{C}$ with the property that $x \in (\mathcal{C}^{\natural})^{\heartsuit}$. Then there exist $y, z \in \mathcal{C}$ such that $x \oplus y \in \mathcal{C}_{\geq 0}$ and $x \oplus z \in \mathcal{C}_{\leq 0}$. Choose decompositions

$$y_{\leq 0} \to y \to y_{\geq 1} \quad \text{and} \quad z_{\leq -1} \to z \to z_{\geq 0}$$

with $y_{\leq 0} \in \mathcal{C}_{\leq 0}$, $z_{\leq -1} \in \mathcal{C}_{\leq -1}$, $y_{\geq 1} \in \mathcal{C}_{\geq 1}$, and $z_{\geq 0} \in \mathcal{C}_{\geq 0}$. Then we obtain cofibre sequences

$$\Sigma^{-1} y_{\geq 1} \to x \oplus y_{\leq 0} \to x \oplus y \quad \text{and} \quad x \oplus z \to x \oplus z_{\geq 0} \to \Sigma z_{\leq -1}$$

showing that $x \oplus y_{\leq 0} \in \mathcal{C}_{\geq 0}$ and $x \oplus z_{\geq 0} \in \mathcal{C}_{\leq 0}$. Since $y_{\leq 0} \in \mathcal{C}_{\leq 0}$, it follows that $x \oplus y_{\leq 0} \oplus z_{\geq 0} \in \mathcal{C}_{\leq 0}$, and similarly $x \oplus y_{\leq 0} \oplus z_{\geq 0} \in \mathcal{C}_{\geq 0}$ because $z_{\geq 0} \in \mathcal{C}_{\geq 0}$. Hence x is a summand of an object in $\mathcal{C}^{\heartsuit}$.

Given an arbitrary $x \in (\mathcal{C}^{\natural})^{\heartsuit}$, choose $y \in \mathcal{C}^{\natural}$ such that $x \oplus y \in \mathcal{C}_{\leq 0}$ and pick a cofibre sequence $y_{\leq -1} \to y \to y_{\geq 0}$ with $y_{\leq -1} \in \mathcal{C}_{\leq 0}$ and $y_{\geq 0} \in (\mathcal{C}_{\geq 0})^{\natural}$. Adding the cofibre sequence $0 \to x \xrightarrow{\mathrm{id}} x$, it follows that $x \oplus y_{\geq 0} \in \mathcal{C}$. Moreover, $x \oplus y_{\geq 0} \in (\mathcal{C}_{\geq 0})^{\natural}$ because $x \in (\mathcal{C}_{\geq 0})^{\natural}$ by assumption. Now pick a cofibre sequence

$$(y_{\geq 0})_{\leq 0} \to y_{\geq 0} \to (y_{\geq 0})_{\geq 1}$$

with $(y_{\geq 0})_{\leq 0} \in (\mathcal{C}_{\leq 0})^{\natural}$ and $(y_{\geq 0})_{\geq 1} \in \mathcal{C}_{\geq 1}$. Then $(y_{\geq 0})_{\leq 0} \in (\mathcal{C}^{\natural})^{\heartsuit}$, and the induced cofibre sequence

$$x \oplus (y_{\geq 0})_{\leq 0} \to x \oplus y_{\geq 0} \to (y_{\geq 0})_{\geq 1}$$

shows that $x \oplus (y_{\geq 0})_{\leq 0} \in \mathcal{C} \cap (\mathcal{C}^{\natural})^{\heartsuit}$. By the preceding paragraph, this implies that $x \oplus (y_{\geq 0})_{\leq 0}$ is a retract of an object in $\mathcal{C}^{\heartsuit}$, which implies that x itself is a retract of an object in $\mathcal{C}^{\heartsuit}$. □

This allows us to generalise [BS01, Theorem 2.8] from exact 1-categories to arbitrary exact categories.

I.2.0.30 Corollary. The exact functors $\mathcal{D}^{\mathrm{b}}(\mathcal{C}^{\natural}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{C})^{\natural}$ induced by the universal property of the stable envelope assemble to a natural equivalence

$$\mathcal{D}^{\mathrm{b}} \circ (-)^{\natural} \xrightarrow{\sim} (-)^{\natural} \circ \mathcal{D}^{\mathrm{b}}.$$

Proof. There is a commutative square

$$\begin{tikzcd}
\mathrm{Cat}^{\mathrm{perf}} \arrow[r] \arrow[d] & \mathrm{Exact}^{\mathrm{perf}} \arrow[d] \\
\mathrm{Cat}^{\mathrm{st}} \arrow[r] & \mathrm{Exact}
\end{tikzcd}$$

in which $\mathrm{Exact}^{\mathrm{perf}}$ denotes the full subcategory of idempotent complete exact categories, and all arrows are the obvious inclusion functors. The bottom horizontal functor admits a left adjoint by Corollary I.2.0.24, so it suffices to show that $\mathcal{D}^{\mathrm{b}}(\mathcal{C})$ is idempotent complete if $\mathcal{C}$ is idempotent complete.

For any exact category $\mathcal{C}$, the induced functor $\mathcal{D}^{\mathrm{b}}(\mathcal{C})^{\heartsuit} \to (\mathcal{D}^{\mathrm{b}}(\mathcal{C})^{\natural})^{\heartsuit}$ is an idempotent completion by Proposition I.2.0.29. Since $\mathcal{C}$ is idempotent complete, it follows from [SW26, Theorem 1.6] that $\mathcal{D}^{\mathrm{b}}(\mathcal{C})^{\heartsuit} \simeq \mathcal{C}$ is idempotent complete, so $\mathcal{D}^{\mathrm{b}}(\mathcal{C})^{\heartsuit} \to (\mathcal{D}^{\mathrm{b}}(\mathcal{C})^{\natural})^{\heartsuit}$ is an equivalence. Invoking [SW26, Theorem 1.6] a second time, it follows that $\mathcal{D}^{\mathrm{b}}(\mathcal{C}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{C})^{\natural}$ is an equivalence as required. □

Chapter I.3

Tensor products of exact categories

In this section, we use the Gabriel–Quillen embedding to show that the internal hom of exact categories admits a left adjoint. The resulting tensor product of exact categories refines to a symmetric monoidal structure on the category of small exact categories. Moreover, we show that both the Gabriel–Quillen embedding and the stable envelope construction admit symmetric monoidal refinements.

Let us first recall the internal hom of exact categories.

I.3.0.1 Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be small exact categories. Define a morphism $\tau\colon F \Rightarrow G$ in $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ to be ingressive if the component $\tau_x\colon F(x) \to G(x)$ is ingressive in $\mathcal{D}$ for all $x \in \mathcal{C}$.

I.3.0.2 Lemma. The pair $(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}), \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})_{\mathrm{in}})$ is an exact category.

Proof. Let $\iota\colon F \Rightarrow G$ be ingressive in $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$. The cofibre $H := \mathrm{cofib}(\iota)$ exists in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, and we have to show that it is exact. If $i\colon x \rightarrowtail y$ is ingressive in $\mathcal{C}$, then both ι_x and $\iota_{\mathrm{cofib}(i)}$ are ingressive, so

[SW26, Lemma 2.4] implies that the square

$$\begin{tikzcd} F(x) \arrow[r, tail, "\iota_x"] \arrow[d, tail, "F(i)"'] & G(x) \arrow[d, tail, "G(i)"] \\ F(y) \arrow[r, tail, "\iota_y"] & G(y) \end{tikzcd}$$

is Reedy cofibrant in the sense that $F(y) \cup_{F(x)} G(x) \to G(y)$ ingressive. Applying [SW26, Lemma 2.4] a second time, it follows that $H(i)\colon H(x) \to H(y)$ is ingressive in $\mathcal{D}$. Since colimits commute, the functor H also preserves pushouts along ingressives, so it is exact. Moreover, the sequence $F \overset{\iota}{\Rightarrow} G \Rightarrow H$ is also a fibre sequence because it is so pointwise.

It follows formally that every span $F_2 \overset{\tau}{\Leftarrow} F_0 \overset{\iota}{\Rightarrow} F_1$ with ι ingressive admits a pushout, that the induced morphism $F_2 \Rightarrow F_2 \cup_{F_0} F_1$ is ingressive, and that the resulting square is also a pullback.

Suppose now that $F_0 \overset{\iota}{\Rightarrow} F_1 \overset{\pi}{\Rightarrow} F_2$ is a cofibre sequence with ι ingressive and let $\tau\colon G \Rightarrow F_2$ be arbitrary. Then the pullback $G \times_{F_2} F_1$ exists and fits into a cofibre sequence $F_0 \overset{\iota'}{\Rightarrow} G \times_{F_2} F_1 \Rightarrow G$ such that ι' is ingressive. Since the category $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ is evidently additive, it follows that $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ is exact. □

I.3.0.3 Remark. The exact structure of Lemma I.3.0.2 has the property that currying induces an equivalence

$$\mathrm{Fun}^{\mathrm{biex}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}_1, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}_2, \mathcal{D})),$$

where the left hand side denotes the category of functors which are exact in both variables.

In particular, for $\mathcal{D}$ Grothendieck prestable we obtain equivalences

$$\begin{aligned} \mathrm{Fun}^{\mathrm{biex}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}) &\simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C}_1), \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C}_2), \mathcal{D})) \\ &\simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C}_1) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{C}_2), \mathcal{D}). \end{aligned}$$

Remark I.3.0.3 already hints at what the construction of the tensor product should be.

I.3.0.4 Construction. Let $\mathcal{C}$ and $\mathcal{D}$ be small exact categories. The composite

$$y\colon \mathcal{C} \times \mathcal{D} \xrightarrow{よ_{\mathrm{lex}} \times よ_{\mathrm{lex}}} \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \times \mathcal{P}_{\mathrm{lex}}(\mathcal{D}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D})$$

defines a biexact functor whose target is Grothendieck prestable by [Lurb, Theorem C.4.2.1].

Denote by $\mathcal{C} \otimes \mathcal{D}$ the smallest extension-closed subcategory of $\mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D})$ containing the essential image of y. By construction, the functor y induces a biexact functor

$$t\colon \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}.$$

The following statement is the main ingredient in showing that this tensor product satisfies the desired universal property.

I.3.0.5 Proposition. The functors

$$I\colon \mathcal{P}_{\mathrm{lex}}(\mathcal{C} \otimes \mathcal{D}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D})$$

induced by the inclusion $i\colon \mathcal{C} \otimes \mathcal{D} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D})$ and the functor

$$T\colon \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C} \otimes \mathcal{D})$$

induced by $よ_{\mathrm{lex}} \circ t\colon \mathcal{C} \times \mathcal{D} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C} \otimes \mathcal{D})$ are mutually inverse equivalences.

Moreover, restriction along t induces an equivalence

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{biex}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

for every exact category $\mathcal{E}$.

Proof. By construction, $I \circ T \simeq \mathrm{id}$ because $y \simeq i \circ t$.

Since T is colimit-preserving and $\mathcal{C} \otimes \mathcal{D}$ is closed under extensions in $\mathcal{P}_{\mathrm{lex}}(\mathcal{C} \otimes \mathcal{D})$, the preimage $T^{-1}(\mathcal{C} \otimes \mathcal{D})$ is an extension-closed subcategory of $\mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D})$. From the definition of $\mathcal{C} \otimes \mathcal{D}$, it follows that $i(\mathcal{C} \otimes \mathcal{D}) \subseteq T^{-1}(\mathcal{C} \otimes \mathcal{D})$, ie $Ti \simeq よ_{\mathrm{lex}} f$ for some endofunctor f of $\mathcal{C} \otimes \mathcal{D}$. Hence

$$i \simeq ITi \simeq I\, よ_{\mathrm{lex}} f \simeq if.$$

Since i is fully faithful, it is a monomorphism in Cat, so $f \simeq \mathrm{id}$. It follows that $Ti \simeq よ_{\mathrm{lex}}$, which implies that $TI \simeq \mathrm{id}$.

Now consider the commutative square

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathrm{biex}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \arrow[d] \\
\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{P}_{\mathrm{lex}}(\mathcal{E})) \arrow[r] & \mathrm{Fun}^{\mathrm{biex}}(\mathcal{C} \times \mathcal{D}, \mathcal{P}_{\mathrm{lex}}(\mathcal{E}))
\end{tikzcd}$$

The vertical arrows are induced by $よ_{\mathrm{lex}}$, and therefore fully faithful. If an exact functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{E})$ restricts to a biexact functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$, its essential image is contained in $\mathcal{E}$ since $\mathcal{E}$ is closed under extensions in $\mathcal{P}_{\mathrm{lex}}(\mathcal{E})$. Hence the above square is a pullback.

The restriction map at the bottom identifies with the equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D}), \mathcal{P}_{\mathrm{lex}}(\mathcal{E})) \simeq \mathrm{Fun}^{\mathrm{biex}}(\mathcal{C} \times \mathcal{D}, \mathcal{P}_{\mathrm{lex}}(\mathcal{E}))$$

of Remark I.3.0.3. □

It is now fairly straightforward to construct a symmetric monoidal structure on Exact.

I.3.0.6 Construction. Let Pair denote the category of small categories $\mathcal{C}$ equipped with a choice of wide subcategory $\mathcal{C}_0$. This category admits finite products, and therefore carries the cartesian symmetric monoidal structure $\mathrm{Pair}^{\times}$.

Define $\mathrm{Exact}^{\otimes}$ to be the suboperad of $\mathrm{Pair}^{\times}$ determined by the following properties:

1. objects in the underlying category are small exact categories;
2. morphisms $(\mathcal{C}_1, \mathcal{C}_{1,\mathrm{in}}) \boxtimes \ldots \boxtimes (\mathcal{C}_n, \mathcal{C}_{n,\mathrm{in}}) \to (\mathcal{C}, \mathcal{C}_{\mathrm{in}})$ correspond to functors $\mathcal{C}_1 \times \ldots \times \mathcal{C}_n \to \mathcal{C}$ which are exact in each variable.

I.3.0.7 Proposition. Let $\mathcal{O}$ be an operad.

1. The operad $\mathrm{Exact}^{\otimes}$ determines a symmetric monoidal structure on Exact with unit object $\mathrm{Proj}(\mathbb{S})$.
2. The functors
$$\mathcal{D}^{\mathrm{b}}_{\geq 0} \colon \mathrm{Exact} \to \mathrm{Cat}^{\mathrm{prest}} \quad \text{and} \quad \mathcal{D}^{\mathrm{b}} \colon \mathrm{Exact} \to \mathrm{Cat}^{\mathrm{st}}$$
refine to symmetric monoidal functors.
3. For every $\mathcal{O}$-monoidal exact category $\mathcal{C}$, the inclusion functor $\mathcal{C} \to \mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C})$ refines to an $\mathcal{O}$-monoidal exact functor such that precomposition with this inclusion induces an equivalence
$$\mathrm{Fun}^{\mathrm{ex}}_{\otimes}(\mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex}}_{\otimes}(\mathcal{C}, \mathcal{D})$$
for every small, prestably $\mathcal{O}$-monoidal category $\mathcal{D}$. The analogous assertion for $\mathcal{D}^{\mathrm{b}}(\mathcal{C})$ and small, stably $\mathcal{O}$-monoidal categories $\mathcal{D}$ holds as well.
4. The functors
$$\mathcal{P}_{\mathrm{lex}} \colon \mathrm{Exact} \to \mathrm{Groth} \quad \text{and} \quad \mathcal{P}^{\mathrm{st}}_{\mathrm{lex}} \colon \mathrm{Exact} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$$
refine to symmetric monoidal functors.

5. For every $\mathcal{O}$-monoidal exact category $\mathcal{C}$, the Yoneda embedding refines to an $\mathcal{O}$-monoidal functor $\mathcal{C} \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ such that precomposition with this embedding induces an equivalence

$$\mathrm{Fun}^{\mathrm{L}}_{\otimes}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex}}_{\otimes}(\mathcal{C}, \mathcal{D}).$$

for every $\mathcal{O}$-monoidal Grothendieck prestable category $\mathcal{D}$. The analogous assertion for $\mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}(\mathcal{C})$ presentably $\mathcal{O}$-monoidal stable categories $\mathcal{D}$ holds as well.

Proof. By construction, Exact is the underlying category of the operad $\mathrm{Exact}^{\otimes}$. Proving that $\mathrm{Exact}^{\otimes}$ is a symmetric monoidal category reduces to showing that the category of biexact functors with source $\mathcal{C} \times \mathcal{D}$ has an initial object, which follows from Proposition I.3.0.5. Since $\mathrm{Proj}(\mathbb{S})$ is the additive category freely generated by a point, it follows that

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C} \otimes \mathrm{Proj}(\mathbb{S}), \mathcal{D}) \simeq \mathrm{Fun}^{\oplus}(\mathrm{Proj}(\mathbb{S}), \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}),$$

which shows that $\mathrm{Proj}(\mathbb{S})$ is a unit object. This proves assertion (1).

By definition, $(\mathrm{Cat}^{\mathrm{prest}})^{\otimes}$ and $(\mathrm{Cat}^{\mathrm{st}})^{\otimes}$ are the full suboperads of $(\mathrm{Cat}^{\mathrm{rex}})^{\otimes}$ spanned by the prestable, respectively stable, categories. In particular, both identify with full suboperads of $\mathrm{Exact}^{\otimes}$, making the inclusion functors $\mathrm{Cat}^{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{prest}} \to \mathrm{Exact}$ lax symmetric monoidal. The left adjoint $\mathrm{SW} \simeq \mathrm{Sp}^{\omega} \otimes -\colon \mathrm{Cat}^{\mathrm{prest}} \to \mathrm{Cat}^{\mathrm{st}}$ canonically refines to a symmetric monoidal functor, and the left adjoint $\mathcal{D}^{\mathrm{b}}_{\geq 0}$ comes equipped with a unique oplax symmetric monoidal structure by [Hau+23, Proposition A]. By comparing universal properties, one finds that $\mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathrm{Proj}(\mathbb{S})) \to \mathrm{Sp}^{\omega}_{\geq 0}$ and $\mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C}_1 \otimes \mathcal{C}_2) \to \mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C}_1) \otimes \mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C}_2)$ are equivalences, so $\mathcal{D}^{\mathrm{b}}_{\geq 0}$ is in fact symmetric monoidal, proving assertion (2).

It follows that the adjunction $\mathcal{D}^{\mathrm{b}}_{\geq 0}\colon \mathrm{Exact} \rightleftarrows \mathrm{Cat}^{\mathrm{prest}} : \mathrm{inc}$ induces an adjunction

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Exact}) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{prest}})$$

for every operad $\mathcal{O}$. Unwinding what this statement means for the unit $\mathcal{C} \to \mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C})$ yields assertion (3). For the statement about $\mathcal{C} \to \mathcal{D}^{\mathrm{b}}(\mathcal{C})$, one uses that the Spanier–Whitehead stabilisation functor $\mathrm{SW} \simeq \mathrm{Sp}^{\omega} \otimes -$ is a symmetric monoidal left adjoint to the inclusion of stable categories into prestable ones.

By [Lura, Proposition 4.8.1.3, Remark 4.8.1.8 & Proposition 4.8.1.15], the ind-completion functor $\mathrm{Ind}\colon \mathrm{Cat}^{\mathrm{rex}} \to \mathrm{Pr}^{\mathrm{L}}$ admits a symmetric monoidal refinement. The lax symmetric monoidal composition

$$\mathrm{Exact} \xrightarrow{\mathcal{D}^{\mathrm{b}}_{\geq 0}} \mathrm{Cat}^{\mathrm{prest}} \xrightarrow{\mathrm{inc}} \mathrm{Cat}^{\mathrm{rex}} \xrightarrow{\mathrm{Ind}} \mathrm{Pr}^{\mathrm{L}}$$

factors over the full suboperad $\mathrm{Groth}^{\otimes}$ of $(\mathrm{Pr}^{\mathrm{L}})^{\otimes}$. Since $\mathrm{Ind} \circ \mathcal{D}^{\mathrm{b}}_{\geq 0} \simeq \mathcal{P}_{\mathrm{lex}}$ by Corollary I.2.0.25, this provides a lax symmetric monoidal refinement of $\mathcal{P}_{\mathrm{lex}}$. The universal property of $\mathcal{P}_{\mathrm{lex}}$ from Theorem I.2.0.7 (5) implies that $\mathcal{P}_{\mathrm{lex}}\colon \mathrm{Exact} \to \mathrm{Groth}$ is symmetric monoidal. The stabilisation functor $\mathrm{Sp}(-) \simeq \mathrm{Sp} \otimes -\colon \mathrm{Groth} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is also symmetric monoidal, so $\mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}$ carries an induced symmetric monoidal structure. This proves assertion (4).

For the last assertion, note that the restriction functors

$$\begin{aligned} \mathrm{Fun}^{\mathrm{L}}_{\otimes}(\mathcal{P}_{\mathrm{lex}}(\mathcal{C}), \mathcal{D}) &\simeq \mathrm{Fun}^{\mathrm{L}}_{\otimes}(\mathrm{Ind}(\mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C})), \mathcal{D}) \\ &\to \mathrm{Fun}^{\mathrm{ex}}_{\otimes}(\mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C}), \mathcal{D}) \\ &\to \mathrm{Fun}^{\mathrm{ex}}_{\otimes}(\mathcal{C}, \mathcal{D}) \end{aligned}$$

are equivalences due to [Lura, Proposition 4.8.1.10] and the version of assertion (4) for large categories. By virtue of Proposition I.2.0.23, the large version of $\mathcal{D}^{\mathrm{b}}_{\geq 0}$ coincides with its small incarnation on small categories, proving the desired universal property.

The analogous claim for $\mathcal{P}^{\mathrm{st}}_{\mathrm{lex}}$ follows since the stabilisation functor $\mathrm{Sp} \otimes -\colon \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is symmetric monoidal. □

For a fixed exact category $\mathcal{D}$, the functor $- \otimes \mathcal{D}$ need not preserve fully faithful functors: for example, $- \otimes \mathrm{Sp}^{\omega} \simeq \mathcal{D}^{\mathrm{b}}(-)$. However, it does preserve inclusions of left special subcategories.

I.3.0.8 Corollary. Let $\mathcal{U}$ be a left special subcategory of a small exact category $\mathcal{C}$. Then for every small exact category $\mathcal{D}$, the induced functor

$$\mathcal{U} \otimes \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$$

exhibits the domain as a left special subcategory.

Proof. Proposition I.2.0.19 asserts that $j_\natural \colon \mathcal{P}_{\mathrm{lex}}(\mathcal{U}) \to \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ is a left adjoint in Pr^{L}. Consequently, the induced functor

$$\mathcal{P}_{\mathrm{lex}}(\mathcal{U}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D}) \xrightarrow{j_\natural \otimes \mathrm{id}} \mathcal{P}_{\mathrm{lex}}(\mathcal{C}) \otimes \mathcal{P}_{\mathrm{lex}}(\mathcal{D})$$

on Lurie tensor products is fully faithful and left adjoint in Pr^{L}. Since $\mathcal{P}_{\mathrm{lex}}$ is symmetric monoidal by Proposition I.3.0.7, this implies that $\mathcal{U} \otimes \mathcal{D}$ is left special in $\mathcal{C} \otimes \mathcal{D}$, again by Proposition I.2.0.19. $\square$

Chapter I.4

The example of synthetic spectra

In this section, we explain in which sense Pstrągowski's categories of synthetic spectra are presentable stable envelopes of exact categories. This was first pointed out to the authors by Robert Burklund.

I.4.0.1 Definition (Pstrągowski [Pst23a, Definition 2.3])**.** An *additive site* is a small site $\mathcal{C}$ such that the underlying category is additive and such that every covering consists of a single morphism.

I.4.0.2 Definition (Pstrągowski [Pst23a, Definition 3.13])**.** Let E be a homotopy associative ring spectrum. We say that a spectrum X is *E-finite projective* if it is finite and E_*X is a finite projective E_*-module. We denote the full subcategory of spectra spanned by E-finite projectives by $\mathrm{Sp}_E^{\mathrm{fp}}$.

I.4.0.3 Example (Pstrągowski [Pst23a, Proposition 3.23])**.** If E is a homotopy associative ring spectrum, one can define a Grothendieck topology on $\mathrm{Sp}_E^{\mathrm{fp}}$ where coverings consist of a single morphism $x \to y$ such that $E_*x \to E_*y$ is surjective. The category $\mathrm{Sp}_E^{\mathrm{fp}}$ of E-finite projectives forms an additive site.

I.4.0.4 Proposition. There is an isomorphism of posets

$$\{\text{coWaldhausen structures on } \mathcal{C}\} \xrightarrow{\sim} \left\{\begin{matrix}\text{Additive quasi-topologies on } \mathcal{C} \\ \text{such that } x \to 0 \text{ is a covering}\end{matrix}\right\}$$

which associates to a coWaldhausen structure the quasi-topology for which a covering is precisely an egressive morphism.

Proof. Every coWaldhausen structure defines a quasi-topology in the specified way because egressive morphisms form a subcategory and are closed under pullbacks along arbitrary maps. Declaring the coverings of an additive quasi-topology to be egressive defines a coWaldhausen structure if $x \to 0$ is a covering for all $x \in \mathcal{C}$. By inspection, these assignments are mutually inverse. □

I.4.0.5 Remark. It is currently unknown to the authors whether there is some (non-tautological) condition one can put on an additive site such that the coWaldhausen structure of Proposition I.4.0.4 is exact.

I.4.0.6 Theorem (Pstrągowski [Pst23a, Theorem 2.8])**.** Let $\mathcal{C}$ be an additive site. An additive presheaf $X \in \mathcal{P}_\Sigma(\mathcal{C})$ is a sheaf if and only if for every fibre sequence $F \to B \to A$ where $B \to A$ is a covering, the induced sequence $X(A) \to X(B) \to X(F)$ is a fibre sequence of anima.

I.4.0.7 Corollary. Let $\mathcal{C}$ be an exact category. A functor $X \in \mathcal{P}_\Sigma(\mathcal{C})$ is a left exact presheaf if and only if it is a sheaf for the egressive topology defined in Proposition I.4.0.4. In particular, we get that $\mathcal{P}_{\mathrm{lex}}(\mathcal{C}) = \mathrm{Shv}_\Sigma(\mathcal{C})$.

I.4.0.8 Remark. The fact that $\mathcal{P}_{\mathrm{lex}}(\mathcal{C}) = \mathrm{Shv}_\Sigma(\mathcal{C})$ was already shown as part of proving Theorem I.2.0.7. We reiterate it here to make the connection to the theory of additive sheaves on additive sites more visible.

I.4.0.9 Definition (Pstrągowski [Pst23a, Definition 3.14]). Let E be a homotopy associative ring spectrum. We say that E is *Adams-type* if

1. E can be written as a filtered colimit $\operatorname{colim}_\alpha E_\alpha \simeq E$ such that

2. every E_α is E-finite projective and the canonical map

$$E^* E_\alpha \to \operatorname{Hom}_{E_*}(E_* E_\alpha, E_*)$$

   is an isomorphism.

I.4.0.10 Definition ([Pst23a, Definition 4.1]). An *E-based synthetic spectrum* is an additive sheaf of spectra on the site $\mathrm{Sp}_E^{\mathrm{fp}}$ with the topology described in Example I.4.0.3. We denote the category of E-based synthetic spectra by Syn_E. If $y\colon \mathrm{Sp}_E^{\mathrm{fp}} \to \mathrm{Shv}_\Sigma(\mathrm{Sp}_E^{\mathrm{fp}})$ denotes the Yoneda embedding into the category of additive sheaves of spaces, then we let $\nu\colon \mathrm{Sp}_E^{\mathrm{fp}} \to \mathrm{Syn}_E$ be the composite of y with

$$\Sigma^\infty\colon \mathrm{Shv}_\Sigma(\mathrm{Sp}_E^{\mathrm{fp}}) \to \mathrm{Shv}_\Sigma^{\mathrm{Sp}}(\mathrm{Sp}_E^{\mathrm{fp}}) =: \mathrm{Syn}_E\,.$$

We name the composite the *synthetic analogue*.

I.4.0.11 Remark. One can show that the synthetic analogue extends to a functor $\nu\colon \mathrm{Sp} \to \mathrm{Syn}_E$ on all spectra. Usually, it is this functor which is called the synthetic analogue. However, for this note only the above restricted version is necessary.

I.4.0.12 Theorem (Pstrągowski [Pst23a, Proposition 4.2]). The category of E-based synthetic spectra is a presentably symmetric monoidal ∞-category generated by the image of the synthetic analogue.

We will now explain how Theorem I.4.0.12 also follows from the results of Chapter I.2.

I.4.0.13 Theorem. Let E be a homotopy associative ring spectrum of Adams-type. Then the coWaldhausen structure determined by the additive site $\mathrm{Sp}_E^{\mathrm{fp}}$ defines an exact category. Furthermore, the synthetic analogue induces a symmetric monoidal equivalence

$$\mathrm{Syn}_E \simeq \mathcal{P}_{\mathrm{lex}}^{\mathrm{st}}(\mathrm{Sp}_E^{\mathrm{fp}}).$$

Proof. To see that the coWaldhausen structure defines an exact structure on $\mathrm{Sp}_E^{\mathrm{fp}}$, it suffices to prove the following:

1. The category $\mathrm{Sp}_E^{\mathrm{fp}}$ is additive.

2. If $f \to x \twoheadrightarrow y$ is a fibre sequence and $x \twoheadrightarrow y$ is egressive, and $f \to z$ is any map in $\mathrm{Sp}_E^{\mathrm{fp}}$, then the pushout $x \oplus_f z$ exists, and there exists an egressive map $x \oplus_f z \twoheadrightarrow w$ such that

$$z \to x \bigoplus_f z \twoheadrightarrow w$$

   is a fibre sequence.

3. If $f \to x \twoheadrightarrow y$ is a fibre sequence and $x \twoheadrightarrow y$ is egressive, then it is also a cofibre sequence.

Property (1) is clear.

Ad (2): Suppose $f \to x \twoheadrightarrow y$ is a fibre sequence and $f \to z$ is any map. We first argue that the pushout $x \oplus_f z$ exists in $\mathrm{Sp}_E^{\mathrm{fp}}$. We know from [Pst23a, Lemma 3.21] that every object in $\mathrm{Sp}_E^{\mathrm{fp}}$ is dualisable, with dual given by the Spanier–Whitehead dual. Spanier–Whitehead duality takes maps which are injective on E-homology to maps which are surjective. In particular, since $Dx \to Df$ is surjective, it follows that $Dx \times_{Df} Dz$ is E-finite projective and thus so is $D(Dx \times_{Df} Dz) \simeq x \oplus_f z$.

Now, consider the diagram

$$\begin{tikzcd} f \arrow[r] \arrow[d] & x \arrow[rr, two heads] \arrow[d] & & y \arrow[d, "\sim"] \\ z \arrow[r] & x \oplus_f z \arrow[rr] & & \mathrm{cof}(z \to x \oplus_f z). \end{tikzcd}$$

Since the composite $x \to x \oplus_f z \to \mathrm{cof}(z \to x \oplus_f z)$ is surjective on E-homology, it follows that so is $x \oplus_f z \to \mathrm{cof}(z \to x \oplus_f z)$. It is clear that

$$z \to x \bigoplus_f z \twoheadrightarrow \mathrm{cof}(z \to x \bigoplus_f z)$$

is a fibre sequence.

Ad (3): Let $f \to x \twoheadrightarrow y$ be a fibre sequence. Note that it is a fibre sequence in Sp, and hence a cofibre sequence Sp. It follows that it is a cofibre sequence in $\mathrm{Sp}_E^{\mathrm{fp}}$.

The synthetic analogue $\nu\colon \mathrm{Sp}_E^{\mathrm{fp}} \to \mathrm{Syn}_E$ refines to a symmetric monoidal functor by [Pst23a, Corollary 2.29 & Lemma 3.21]. Using [Pst23a, Proposition 4.2] and Proposition I.3.0.7 (5) provides a symmetric monoidal, colimit-preserving functor

$$\mathcal{P}_{\mathrm{lex}}^{\mathrm{st}}(\mathrm{Sp}_E^{\mathrm{fp}}) \to \mathrm{Syn}_E\,.$$

Due to Theorem I.4.0.6, the underlying functor is an equivalence. □

I.4.0.14 Remark. As a result of Theorem I.4.0.13, Theorem I.4.0.12 may be regarded as a special case of Theorem I.2.0.11 and Proposition I.3.0.7.

I.4.0.15 Remark. It also follows from [Pst23a, Lemma 4.23] that $\mathrm{Sp}_E^{\mathrm{fp}}$ forms an extension-closed subcategory of Syn_E. This provides an alternate proof of Theorem I.4.0.13, but would make the entire discussion circular.

Chapter I.5

Localising invariants

As another application of our results, we will now introduce a fairly natural notion of localising invariants for exact categories and show that it passes the obvious sanity checks: every localising invariant valued in connective spectra deloops uniquely to spectrum-valued localising invariant, and algebraic K-theory is the universal localising invariant under the groupoid core.

I.5.0.1 Definition. Call a reduced functor $F\colon \mathrm{Exact} \to \mathcal{X}$

1. *left localising* if for every left special subcategory inclusion $\mathcal{U} \subseteq \mathcal{C}$ the induced square

$$\begin{tikzcd} F(\mathcal{U}) \arrow[r] \arrow[d] & F(\mathcal{C}) \arrow[d] \\ * \simeq F(0) \arrow[r] & F(\mathcal{C}/\mathcal{U}) \end{tikzcd}$$

   is a pullback.

2. *right localising* if $F \circ (-)^{\mathrm{op}}\colon \mathrm{Exact} \to \mathcal{X}$ is left localising.

Denote by $\mathrm{Fun}^{\mathrm{lloc}}(\mathrm{Exact}, \mathcal{X})$ and $\mathrm{Fun}^{\mathrm{rloc}}(\mathrm{Exact}, \mathcal{X})$ the full subcategories of the functor category $\mathrm{Fun}(\mathrm{Exact}, \mathcal{X})$ spanned by the left localising and right localising functors, respectively.

I.5.0.2 Remark. Since $\mathcal{C} \to \mathcal{C}^\natural$ is the inclusion of a left special subcategory, it follows that left (or right) localising functors are invariant under idempotent completion.

I.5.0.3 Lemma. Let $\mathcal{U}$ be a left special subcategory of an exact category $\mathcal{C}$. Then

$$\mathcal{D}^{\mathrm{b}}(\mathcal{U})^\natural \to \mathcal{D}^{\mathrm{b}}(\mathcal{C})^\natural \to \mathcal{D}^{\mathrm{b}}(\mathcal{C}/\mathcal{U})^\natural$$

is a Karoubi sequence of stable categories.

Proof. The induced functor $\mathcal{D}^{\mathrm{b}}(\mathcal{U}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{C})$ is also fully faithful by Remark I.2.0.22. Since $\mathcal{D}^{\mathrm{b}} \colon \mathrm{Exact} \to \mathrm{Cat}^{\mathrm{st}}$ is left adjoint by Corollary I.2.0.24,

$$\mathcal{D}^{\mathrm{b}}(\mathcal{U}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{C}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{C}/\mathcal{U})$$

is a cofibre sequence of stable categories. Taking idempotent completions yields a Karoubi sequence. □

I.5.0.4 Corollary. The functor

$$\mathrm{K} \circ (-)^\natural \colon \mathrm{Exact} \to \mathrm{Sp}_{\geq 0}$$

is both left and right localising.

Proof. Since K-theory is invariant under taking opposites (see eg [Bar15, Corollary 5.16.1]), it suffices to prove that $\mathrm{K} \circ (-)^\natural$ is left localising.

The localisation theorem for stable categories and Lemma I.5.0.3 together imply that

$$\mathrm{K}(\mathcal{D}^{\mathrm{b}}(\mathcal{U})^\natural) \to \mathrm{K}(\mathcal{D}^{\mathrm{b}}(\mathcal{C})^\natural) \to \mathrm{K}(\mathcal{D}^{\mathrm{b}}(\mathcal{C}/\mathcal{U})^\natural)$$

is a fibre sequence of connective spectra. Since $\mathcal{D}^{\mathrm{b}}$ commutes with idempotent completion by Corollary I.2.0.30, the Gillet–Waldhausen theorem in connective K-theory [SW26, Theorem 1.7] implies that $\mathrm{K} \circ (-)^{\natural}$ is left localising. □

We are now able to prove the following variant of [CK, Theorem 4.4.3]. Note that our argument differs from *loc. cit.* because instead of the Bass delooping we deloop by tensoring with the "Calkin category" of the sphere. While we consider this construction to be well-known to the experts (at least in the case of stable categories), we are not aware of a citable source in the literature for this type of argument.

I.5.0.5 Theorem.

1. If $\mathcal{X}$ is a pointed and finitely complete category, then the categories $\mathrm{Fun}^{\mathrm{lloc}}(\mathrm{Exact}, \mathcal{X})$ and $\mathrm{Fun}^{\mathrm{rloc}}(\mathrm{Exact}, \mathcal{X})$ are stable. If $\mathcal{X}$ admits filtered colimits and these commute with finite limits, the same is true for the full subcategories of finitary functors.

2. If $\mathcal{Y}$ is a stable category with a right complete t-structure, then taking connective covers induces equivalences

$$\tau_{\geq 0} \colon \mathrm{Fun}^{\mathrm{lloc}}(\mathrm{Exact}, \mathcal{Y}) \to \mathrm{Fun}^{\mathrm{lloc}}(\mathrm{Exact}, \mathcal{Y}_{\geq 0})$$

   and

$$\tau_{\geq 0} \colon \mathrm{Fun}^{\mathrm{rloc}}(\mathrm{Exact}, \mathcal{Y}) \to \mathrm{Fun}^{\mathrm{rloc}}(\mathrm{Exact}, \mathcal{Y}_{\geq 0}).$$

   If $\mathcal{Y}_{\leq 0}$ is closed under filtered colimits, the same is true for the subcategories of finitary functors.

Proof. Precomposition with $(-)^{\mathrm{op}}$ induces an equivalence between the categories of left localising and right localising functors, so we only have to consider left localising functors.

Since $\mathrm{Fun}^{\mathrm{lloc}}(\mathrm{Exact}, \mathcal{X}_{\geq 0})$ is pointed and has finite limits (which are computed pointwise), stability follows once we show that Ω is invertible.

Let $\mathcal{F}$ be the additive category of countably generated projective $\mathbb{S}$-modules. This contains the category $\mathrm{Proj}(\mathbb{S})$ of finitely generated free $\mathbb{S}$-modules as a full additive subcategory. Set $\mathcal{Q} := \mathcal{F}/\operatorname{Proj}(\mathbb{S})$. By Corollary I.3.0.8, we obtain for every left localising functor a natural fibre sequence

$$F(-) \to F(- \otimes \mathcal{F}) \to F(- \otimes \mathcal{Q})$$

because the tensor product commutes with colimits in both variables. Since $\mathcal{F}$ admits a functorial Eilenberg swindle, the same is true for $\mathcal{C} \otimes \mathcal{F}$ for any exact category $\mathcal{C}$. Hence $\Omega F(- \otimes \mathcal{Q}) \simeq F$. Since precomposition with $- \otimes \mathcal{Q}$ and postcomposition with Ω commute, this shows that Ω has both a left and a right inverse.

If filtered colimits commute with finite limits, then the loops functor on finitary, left localising functors is still given by pointwise application of Ω. Since $- \otimes \mathcal{Q}$ commutes with colimits, it follows that the finitary, left localising functors form a stable subcategory of all left localising functors.

The statement about left localising functors $\mathrm{Exact} \to \mathcal{Y}$ is immediate from the stability of $\mathrm{Fun}^{\mathrm{lloc}}(\mathrm{Exact}, \mathcal{Y}_{\geq 0})$ since

$$\begin{aligned}\mathrm{Fun}^{\mathrm{lloc}}(\mathrm{Exact}, \mathcal{Y}) &\simeq \mathrm{Fun}^{\mathrm{lloc}}(\mathrm{Exact}, \lim_{\Omega} \mathcal{Y}_{\geq 0}) \\ &\simeq \lim_{\Omega} \mathrm{Fun}^{\mathrm{lloc}}(\mathrm{Exact}, \mathcal{Y}_{\geq 0}),\end{aligned}$$

where $\lim_{\Omega}$ denotes the sequential limit over iterates of the loops functor. Moreover, observe that any left localising functor $F\colon \mathrm{Exact} \to \mathcal{Y}$ satisfies

$$F \simeq \operatorname*{colim}_n \tau_{\geq -n} F \simeq \operatorname*{colim}_n \Omega^n \tau_{\geq 0} \Sigma^n F \simeq \operatorname*{colim}_n \Omega^n \tau_{\geq 0} F(- \otimes \mathcal{Q}^{\otimes n})$$

by right completeness of the t-structure and the preceding discussion. If $\tau_{\geq 0}$ preserves filtered colimits, it follows that F is finitary if and only if $\tau_{\geq 0}F$ is finitary. $\square$

I.5.0.6 Remark. Let $\mathcal{X}$ be a stable category with a right complete t-structure and let $F\colon \mathrm{Exact} \to \mathcal{X}_{\geq 0}$ be a left localising functor. Suppose that there exists a natural equivalence $F \simeq F \circ (-)^{\mathrm{op}}$. Then F is both left and right localising, so there exist two a priori different deloopings F^{lloc} and F^{rloc} of F as a left localising and right localising functor, respectively.

However, $F^{\mathrm{rloc}} \circ (-)^{\mathrm{op}}$ is by definition a left localising invariant satisfying

$$\tau_{\geq 0} \circ F^{\mathrm{rloc}} \circ (-)^{\mathrm{op}} \simeq F \circ (-)^{\mathrm{op}} \simeq F,$$

so $F^{\mathrm{rloc}} \simeq F^{\mathrm{lloc}}$. In particular, such a functor admits a unique delooping which is both left and right localising and is invariant under taking opposite categories. Note, however, that this identification does depend on the choice of equivalence $F \simeq F \circ (-)^{\mathrm{op}}$.

I.5.0.7 Definition. Define the non-connective algebraic K-theory functor

$$\mathbb{K}\colon \mathrm{Exact} \to \mathrm{Sp}$$

as the (left or right) localising functor satisfying $\tau_{\geq 0}\mathbb{K} \simeq \mathrm{K} \circ (-)^{\natural}$.

I.5.0.8 Proposition.

1. The functor $\mathbb{K}$ coincides with Schlichting's non-connective algebraic K-theory functor on exact 1-categories introduced in [Sch04].

2. The functor $\mathbb{K}$ coincides with the non-connective algebraic K-theory functor on stable categories defined by Blumberg, Gepner and Tabuada in [BGT13a].

Proof. The proof that Schlichting's delooping coincides with the canonical one amounts to an alternative argument for Theorem I.5.0.5 using a generalisation of Schlichting's construction.

Let $\mathcal{C}$ be an exact category. Define $\mathcal{F}\mathcal{C} \subseteq \mathcal{P}_{\mathrm{lex}}(\mathcal{C})$ as the full subcategory on those objects which are equivalent to a sequential colimit

$$よ_{\mathrm{lex}}(x_0) \to よ_{\mathrm{lex}}(x_1) \to よ_{\mathrm{lex}}(x_2) \to \dots$$

along (images of) ingressives in $\mathcal{C}$. In the following, we will omit the Yoneda embedding $よ_{\mathrm{lex}}$ from notation.

To show that $\mathcal{F}\mathcal{C}$ is extension-closed, consider a cofibre sequence $X \to Y \xrightarrow{p} Z$ and choose diagrams $x_0 \rightarrowtail x_1 \rightarrowtail x_2 \rightarrowtail \dots$ and $z_0 \to z_1 \rightarrowtail z_2 \rightarrowtail \dots$ whose colimits are X and Z, respectively. Pulling back the filtration of Z along p, we obtain a diagram

$$Y \times_Z z_0 \to Y \times_Z z_1 \to Y \times_Z z_2 \to \dots$$

whose colimit is Y. For fixed n, we have a cofibre sequence

$$X \to Y \times_Z z_n \to z_n.$$

Since $\mathcal{C}$ is left special in $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$, there exists a pushout

$$\begin{tikzcd} y'_n \arrow[r, tail] \arrow[d] & y_n \arrow[d] \\ X \arrow[r] & Y \times_Z z_n \end{tikzcd}$$

with $y'_n \rightarrowtail y_n$ ingressive in $\mathcal{C}$. Since $X \simeq \operatorname{colim}_n x_n$, the morphism $y'_n \to X$ factors over some x_{k_n}, and we conclude that

$$Y \times_Z z_n \simeq \operatorname*{colim}_{k \geq k_n} \left(x_k \cup_{y'_n} y_n \right).$$

Without loss of generality, we may assume that the sequence $(k_n)_n$ is strictly increasing. This presents Y as the colimit of a diagram

$$\operatorname*{colim}_{k \geq k_0} \big(x_k \cup_{y'_0} y_0\big) \to \operatorname*{colim}_{k \geq k_1} \big(x_k \cup_{y'_1} y_1\big) \to \operatorname*{colim}_{k \geq k_2} \big(x_k \cup_{y'_2} y_2\big) \to \dots$$

We define another sequence $(l_n)_n$ by induction. Set $l_0 := k_0$. Assuming l_n has been defined, choose $l_{n+1} > l_n$ such that

$$\begin{aligned} l_{n+1} > \min\{k \mid k \geq k_{n+1} \text{ and } x_{l_n} \cup_{y'_n} y_n \to \operatorname*{colim}_{k \geq k_{n+1}} (x_k \cup_{y'_{n+1}} y_{n+1}) \\ \text{factors over } x_{l_{n+1}} \cup_{y'_{n+1}} y_{n+1}\}. \end{aligned}$$

Then the objects $a_n := x_{l_n} \cup_{y'_n} y_n$ assemble into a directed system

$$a_0 \to a_1 \to a_2 \to \dots$$

Each connecting map in this system fits into a diagram of exact sequences

$$\begin{tikzcd} x_{l_n} \arrow[r, tail] \arrow[d, tail] & a_n \arrow[r, two heads] \arrow[d] & z_n \arrow[d, tail] \\ x_{l_{n+1}} \arrow[r, tail] & a_{n+1} \arrow[r, two heads] & z_{n+1} \end{tikzcd}$$

Since the two outer vertical arrows are ingressive as indicated, the middle vertical is also ingressive. Taking the colimit of these diagrams over n, we obtain a cofibre sequence

$$\operatorname*{colim}_n x_{l_n} \to \operatorname*{colim}_n a_n \to \operatorname*{colim}_n z_n$$

which maps to the exact sequence $X \to Y \to Z$. Since this transformation is an equivalence on the two outer terms, it follows that $Y \simeq \operatorname{colim}_n a_n$. Hence $\mathcal{FC}$ is an extension-closed subcategory of $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$.

Since $\mathcal{C}$ is left special in $\mathcal{P}_{\mathrm{lex}}(\mathcal{C})$, it is also left special in $\mathcal{FC}$. Denote by $\mathcal{QC} := \mathcal{FC}/\mathcal{C}$ the quotient exact category. Then we obtain a functorial fibre sequence

$$\mathrm{K}(\mathcal{C}^\natural) \to \mathrm{K}((\mathcal{FC})^\natural) \to \mathrm{K}((\mathcal{QC})^\natural)$$

by Corollary I.5.0.4. As in the proof of Theorem I.5.0.5, it follows that

$$\mathbb{K}(\mathcal{C}) \simeq \operatorname*{colim}_n \Omega^n \operatorname{K}((\mathcal{Q}^n\mathcal{C})^\natural) \simeq \operatorname*{colim}_n \Omega^n \operatorname{K}(\mathcal{Q}^n\mathcal{C})),$$

where the second equivalence is a consequence of the cofinality theorem [Bar16, Theorem 10.11].

In the case of an exact 1-category, $\mathcal{F}\mathcal{C}$ is exactly the "countable envelope" used in [Sch04, Section 3], see also [Kel90, Appendix B]. It follows from [Sch04, Lemma 3.2] and [Win, Theorem 3.12] that the quotient exact category $\mathcal{Q}$ is an exact 1-category, so this construction reproduces precisely Schlichting's definition of non-connective algebraic K-theory in [Sch04].

For $\mathcal{C}$ a stable category, $\mathcal{F}\mathcal{C}$ coincides with the full subcategory of $\aleph_1$-compact objects in $\operatorname{Ind}(\mathcal{C})$ because every countable filtered diagram receives a cofinal functor from the poset $\mathbb{N}$. Hence, this construction simultaneously reproduces the construction of non-connective algebraic K-theory from [BGT13a, Section 9]. □

I.5.0.9 Corollary. Non-connective algebraic K-theory is the initial left localising, Sp-valued invariant under $\Sigma^\infty\iota$.

Proof. Observing that every left localising invariant is in particular additive, this follows from Theorem I.5.0.5 and the universality of K (see eg [HLS23, Theorem 5.1]): for every left localising invariant $F\colon \mathrm{Exact} \to \mathrm{Sp}$, we have

$$\operatorname{Nat}(\mathbb{K}, F) \simeq \operatorname{Nat}(\operatorname{K}, \tau_{\geq 0}F) \simeq \operatorname{Nat}(\Sigma^\infty\iota, \tau_{\geq 0}F) \simeq \operatorname{Nat}(\Sigma^\infty\iota, F). \quad \square$$

I.5.0.10 Corollary (Gillet–Waldhausen)**.** Let $\mathcal{X}$ be a stable category with a right complete t-structure and let $F\colon \mathrm{Exact} \to \mathcal{X}_{\geq 0}$ be a left localising invariant.

1. If the unit transformation $\mathcal{C} \to \mathcal{D}^{\mathrm{b}}(\mathcal{C})$ induces an equivalence $F(\mathcal{C}) \to F(\mathcal{D}^{\mathrm{b}}(\mathcal{C}))$ for all exact categories $\mathcal{C}$, then the same is true for the non-connective delooping of F.

2. Assume in addition that $\mathcal{X}_{\leq 0}$ is closed under filtered colimits and that F is finitary. Then $F(\mathcal{C}) \to F(\mathcal{D}^{\mathrm{b}}(\mathcal{C}))$ is an equivalence for every small exact category $\mathcal{C}$.

3. The unit transformation induces an equivalence

$$\mathbb{K}(\mathcal{C}) \xrightarrow{\sim} \mathbb{K}(\mathcal{D}^{\mathrm{b}}(\mathcal{C}))$$

for every exact category $\mathcal{C}$.

Proof. Lemma I.5.0.3 and Remark I.5.0.2 imply that $F \circ \mathcal{D}^{\mathrm{b}}$ is left localising, so the first assertion follows by observing that taking connective covers of left localising $\mathcal{X}$-valued functors is conservative by Theorem I.5.0.5.

For the second assertion, the inclusion $\mathcal{C} \to \mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C})$ is left special by Lemma I.2.0.9. Since $\mathcal{D}^{\mathrm{b}}_{\geq 0}\mathcal{C}$ is generated by $\mathcal{C}$ under finite colimits, the quotient of $\mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C})$ by $\mathcal{C}$ is trivial. It follows that $F(\mathcal{C}) \to F(\mathcal{D}^{\mathrm{b}}_{\geq 0}(\mathcal{C}))$ is an equivalence because F is left localising. Moreover, F is invariant under Spanier–Whitehead stabilisation because it preserves filtered colimits.

The assertion about algebraic K-theory follows by combining the first and second assertion with Corollary I.5.0.4. $\square$

I.5.0.11 Corollary. The non-connective algebraic K-theory functor

$$\mathbb{K} \colon \mathrm{Exact} \to \mathrm{Sp}$$

admits a lax symmetric monoidal refinement.

Proof. Non-connective algebraic K-theory is known to refine to a lax symmetric monoidal functor $\mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ by [BGT14, Corollary 1.6]; note that *loc. cit.* speaks about the same functor as we do by Proposition I.5.0.8. Since $\mathcal{D}^{\mathrm{b}}$ is symmetric monoidal by Proposition I.3.0.7 (2), the corollary follows from Corollary I.5.0.10. $\square$

Part II

The formal spectrum of a tensor-triangulated category

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Abstract. To any essentially small tensor-triangulated category $\mathcal{K}$ and Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{K})$ we associate a ringed space $(\mathrm{Spf}(\mathcal{K}, Y), \mathcal{O}_{\mathrm{Spf}(\mathcal{K},Y)})$, called the *formal spectrum* of $(\mathcal{K}, Y)$. We establish basic properties of this construction and compute it in several examples from algebraic geometry, chromatic homotopy theory, equivariant homotopy theory, and modular representation theory.

Contents

Chapter II.1

Introduction

Tensor–triangular geometry packages a tensor–triangulated category into a geometric object via the Balmer spectrum [Bal05]. Concretely, if $\mathcal{K}$ is an essentially small tt-category (or a 2-*ring* in our terminology), then its Balmer spectrum $\mathrm{Spc}(\mathcal{K})$ is a spectral space. Its topology has a basis of closed sets given by supports of objects in $\mathcal{K}$, and it carries a natural structure sheaf $\mathcal{O}_{\mathrm{Spc}(\mathcal{K})}$ valued in graded rings. When $\mathcal{K} = \mathrm{D}_{\mathrm{qc}}(X)^c$ for a quasi-compact and quasi-separated (qcqs) scheme X, one recovers X as a locally ringed space from $(\mathrm{Spc}(\mathcal{K}), \mathcal{O}_{\mathrm{Spc}(\mathcal{K})})$; see Example II.3.0.20. This is Balmer's reconstruction theorem [Bal02], relying crucially on Thomason's classification of thick tensor ideals in $\mathrm{D}_{\mathrm{qc}}(X)^c$ [Tho97].

In classical algebraic geometry, passing from a scheme X to its formal completion $\widehat{X}_Y$ along a closed subset Y isolates the infinitesimal neighborhood of Y and remembers all nilpotent thickenings at once. This is preferable whenever a problem is intrinsically local around Y, and it is natural to ask for an analogue in tensor–triangular geometry. For example, the $K(n)$-local stable homotopy category is obtained from the $E(n)$-local category by a process resembling adic completion (see the formulas in [HS99, Proposition 7.10]), and it morally behaves like a "residue field". One might therefore expect its spectrum to be a single point. However, in tt-geometry one must work with essentially small subcategories, and the compact objects in the $K(n)$-local category do not contain the unit (hence do not form a tt-category), while the spectrum of dualizable objects is, perhaps, unexpectedly large [BHN22].

The goal of this paper is therefore to develop a formal analogue of the Balmer spectrum. Given an essentially small tt-category[1] $\mathcal{K}$ and a Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{K})$, we define the *formal spectrum* $\mathrm{Spf}(\mathcal{K}, Y)$ to be a ringed space whose underlying topological space is Y, equipped with a completed analogue of Balmer's structure sheaf; see Definition II.5.0.1 for the complete definition. Roughly speaking, $\mathrm{Spc}(\mathcal{K})$ plays the role of $\mathrm{Spec}(R)$, and $\mathrm{Spf}(\mathcal{K}, Y)$ is an analogue of the formal spectrum $\mathrm{Spf}(R, I)$ of a ring R along an ideal I (see Definition II.2.0.1 for our conventions on formal spectra of rings, which differ slightly from some standard sources).

In the classical case, it is straightforward to check that the completion map $R \to R_I^\wedge$ induces an isomorphism of formal schemes

$$\mathrm{Spf}(R_I^\wedge, IR_I^\wedge) \xrightarrow{\sim} \mathrm{Spf}(R, I).$$

An analogue holds in the tt-world, although the proof is more delicate. Since Bousfield localization is most naturally formulated in the presentable setting, it is convenient to pass from an essentially small $\mathcal{K}$ to the associated stable homotopy theory $\mathcal{C} := \mathrm{Ind}(\mathcal{K})$. Inside $\mathcal{C}$ we can form the completion at $Y \subseteq \mathrm{Spc}(\mathcal{K})$, obtaining a full subcategory $\widehat{\mathcal{C}}_Y$ of Y-complete objects together with a symmetric monoidal localization $\widehat{(-)}_Y \colon \mathcal{C} \to \widehat{\mathcal{C}}_Y$. Unlike categories usually considered in tensor–triangular geometry, $\widehat{\mathcal{C}}_Y$ is typically not rigidly-compactly generated. Our first main result identifies the formal spectrum computed

[1] For technical reasons, we work with stable ∞-categories throughout; see Remark II.3.0.3.

from the dualizable objects of $\widehat{\mathcal{C}}_Y$ with the original definition in terms of $(\mathcal{K}, Y)$.

Theorem A (Theorem II.5.0.14)**.** *Let $\mathcal{K}$ be a 2-ring and let $Y \subseteq \operatorname{Spc}(\mathcal{K})$ be a Thomason subset. There is a natural isomorphism of ringed spaces*

$$(\varphi, \varphi^{\#})\colon\ \operatorname{Spf}(\widehat{\mathcal{C}}^{\mathrm{d}}_Y, \varphi^{-1}(Y)) \longrightarrow \operatorname{Spf}(\mathcal{K}, Y),$$

where $\varphi = \operatorname{Spc}(\widehat{(-)}_Y)$ is the map on Balmer spectra induced by completion.

This result is fundamental for us: the left-hand side is often the conceptual object of interest, while the right-hand side is typically easier to compute. For the remainder of this introduction, we abbreviate the left-hand side to $\operatorname{Spf}(\widehat{\mathcal{C}}^{\mathrm{d}}_Y)$, suppressing the implicit Thomason subset $\varphi^{-1}(Y)$.

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A basic computation in tt-geometry is the Hopkins–Neeman theorem: for any commutative noetherian ring R one has a canonical identification $\operatorname{Spc}(\mathrm{D}(R)^c) \cong \operatorname{Spec}(R)$, under which tt-support agrees with ordinary (homological) support. Since formal completion in algebraic geometry is taken along a closed subset, we fix an ideal $I \subseteq R$ and consider the closed subset $V(I) \subseteq \operatorname{Spec}(R) \cong \operatorname{Spc}(\mathrm{D}(R)^c)$. Our formal spectrum recovers the usual affine formal scheme:

Theorem B (Theorem II.6.0.3)**.** *Let R be a commutative noetherian ring and $I \subseteq R$ an ideal. There is a natural isomorphism of formal schemes*

$$\rho\colon\ \operatorname{Spf}\big(\mathrm{D}(R)^c, V(I)\big) \xrightarrow{\sim} \operatorname{Spf}(R, I).$$

Consequently, the completion of $\mathrm{D}(R)$ at $V(I)$ satisfies

$$\operatorname{Spf}\big(\widehat{\mathrm{D}(R)}^{d}_{V(I)}\big) \xrightarrow{\sim} \operatorname{Spf}(R^{\wedge}_I).$$

The completed category $\widehat{\mathrm{D}(R)}_{V(I)}$ here is the usual complete derived category of a noetherian ring at an ideal; see Example II.4.0.6.

We next extend this affine computation to noetherian schemes. The key input is a locality statement for the formal spectra with respect to restriction to quasi-compact opens. If $U \subseteq \mathrm{Spc}(\mathcal{K})$ is quasi-compact with closed complement Z, recall that $\mathcal{K}(U) \coloneqq (\mathcal{K}/\mathcal{K}_Z)^\natural$ denotes the idempotent-completed Verdier quotient (cf. Remark II.3.0.10). Then we prove:

Theorem C (Theorem II.5.0.16)**.** *Let $\mathcal{K}$ be a 2-ring and $Y \subseteq \mathrm{Spc}(\mathcal{K})$ a Thomason subset. Let $U \subseteq \mathrm{Spc}(\mathcal{K})$ be a quasi-compact open subset and let $q_U \colon \mathcal{K} \to \mathcal{K}(U)$ be the canonical functor. Set $Y' \coloneqq Y \cap U$, viewed as a Thomason subset of $\mathrm{Spc}(\mathcal{K}(U)) \cong U$. Then restriction along $Y' \hookrightarrow Y$ induces an isomorphism of ringed spaces*

$$\mathrm{Spf}(\mathcal{K}(U), Y') \xrightarrow{\cong} \mathrm{Spf}(\mathcal{K}, Y)\big|_{Y'}.$$

With Theorem C in hand, the affine computation globalizes by covering a noetherian scheme by finitely many affine opens and checking that the resulting local identifications glue on overlaps. This yields a formal version of Thomason's theorem.

Theorem D (Theorem II.7.0.7)**.** *Let X be a noetherian scheme, and let $Z \subseteq X$ be a closed subset. Then there is a natural isomorphism of formal schemes*

$$\Phi \colon \mathrm{Spf}\big(\mathrm{D_{qc}}(X)^c, Z\big) \xrightarrow{\sim} \widehat{X}_Z,$$

where $\widehat{X}_Z$ denotes the formal completion of X along Z.

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Our second main family of examples comes from chromatic homotopy theory. Fix a prime p and a height $n \geq 0$, and write $\mathcal{S}p_n$ for the

$E(n)$-local stable homotopy category with compact objects $\mathcal{S}p_n^c$. By work of Hovey–Strickland, the Balmer spectrum $\mathrm{Spc}(\mathcal{S}p_n^c)$ is the finite chain of primes

$$\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \cdots \subsetneq \mathcal{P}_n$$

indexed by chromatic height, and the structure sheaf on the basic opens can be described in terms of the localizations $L_k S^0$; see Theorem II.9.0.5. For each $0 \le h \le n-1$ we consider the Thomason subset

$$Y_{h+1} = \{\mathcal{P}_{h+1}, \ldots, \mathcal{P}_n\} \subseteq \mathrm{Spc}(\mathcal{S}p_n^c),$$

corresponding to spectra of type at least $h+1$. Completion at Y_{h+1} recovers the local category $\mathcal{S}p_{K(h+1)\vee\cdots\vee K(n)}$, and we compute the following:

Theorem E (Theorem II.9.0.8)**.** *Fix an integer $0 \le h \le n-1$, then the formal spectrum $\mathrm{Spf}\big(\mathcal{S}p^{\mathrm{d}}_{K(h+1)\vee\cdots\vee K(n)}\big)$ has underlying space Y_{h+1}, and its structure sheaf on basic opens is given by*

$$\mathcal{O}(\overline{U}_k) \cong \pi_* L_{K(h+1)\vee\cdots\vee K(k)} S^0 \qquad (k \ge h+1),$$

for the opens $\overline{U}_k = U(L_n F(k+1)) \cap Y_{h+1}$ described in Theorem II.9.0.5.

These chromatic examples illustrate one of the motivations for introducing $\mathrm{Spf}(\mathcal{K}, Y)$: completion can drastically change the geometry detected by dualizable objects, and the formal spectrum provides a convenient receptacle for the resulting local information. In particular, in the special case where $h = n-1$, we see that the formal spectrum of dualizable objects in the $K(n)$-local category is indeed a single point, as one would hope. We conclude with two further examples, one from equivariant homotopy theory and one from modular representation theory, to illustrate additional behavior of the formal spectrum.

Organization of the paper. In Chapter II.2 we recall formal schemes and formal completions in algebraic geometry and fix conventions. In

Chapter II.3 we review the Balmer spectrum and its structure sheaf. In Chapter II.4 we recall completion of stable homotopy theories at Thomason subsets, and in Chapter II.5 we introduce the formal spectrum $\mathrm{Spf}(\mathcal{K}, Y)$ and prove the completion theorem Theorem II.5.0.14. In Chapters II.6 and II.7 we establish the formal version of the Hopkins–Neeman and Thomason theorems. Chapter II.8 produces a comparison map, analogous to Balmer's for the ordinary spectrum and in Chapter II.9 we compute the formal spectra arising in chromatic homotopy theory. Finally, we finish in Chapter II.10 with the two examples from equivariant homotopy theory and modular representation theory.

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Chapter II.2

Formal schemes and formal completions

We begin by recalling the definition of the formal spectrum of a commutative noetherian ring. We do this to fix our conventions, which may differ slightly from the standard literature.

II.2.0.1 Definition. Let R be a commutative noetherian ring and let $I \subseteq R$ be an ideal. The *formal spectrum* of R (with respect to I), denoted $\operatorname{Spf}(R, I)$, is the locally ringed space

$$(|\operatorname{Spf}(R, I)|, \mathcal{O}_{\operatorname{Spf}(R,I)})$$

where:

(a) The underlying topological space $|\operatorname{Spf}(R, I)|$ is the closed subset of the Zariski spectrum given by the vanishing locus of I, i.e.,

$$|\operatorname{Spf}(R, I)| := V(I) = \left\{ \mathfrak{p} \in \operatorname{Spec}(R) \,\middle|\, I \subseteq \mathfrak{p} \right\}.$$

It has a basis of open subsets of the form

$$\overline{D}(s) := D(s) \cap V(I) = \left\{ \mathfrak{p} \in \operatorname{Spec}(R) \,\middle|\, I \subseteq \mathfrak{p},\ s \notin \mathfrak{p} \right\},$$

where $s \in R$ and

$$D(s) = \left\{ \mathfrak{p} \in \mathrm{Spec}(R) \,\middle|\, s \notin \mathfrak{p} \right\}.$$

(b) The structure sheaf $\mathcal{O}_{\mathrm{Spf}(R,I)}$ is defined on the basic open sets $\overline{D}(s)$ as the I-adic completion of the localization $R[1/s]$.

II.2.0.2 Example. Let R be a noetherian ring, and let I be a nilpotent ideal (for example, $I = (0)$). Then it is immediate from the definitions that

$$\mathrm{Spf}(R, I) \cong \mathrm{Spec}(R),$$

the usual Zariski spectrum.

II.2.0.3 Example. Let $R = \mathbb{Z}$ and $I = (p)$ for a prime p. Then $\mathrm{Spf}(\mathbb{Z}, (p))$ is a one-point locally ringed space with ring of global sections $\mathbb{Z}_p$.

II.2.0.4 Remark. There is an obvious extension to $\mathbb{Z}$-graded commutative rings, where we work with *homogeneous* prime ideals. We denote this by a superscript h, i.e., $\mathrm{Spec}^h(R_*)$ denotes the homogeneous Zariski spectrum of a graded ring R_*, and $\mathrm{Spf}^h(R, I)$ denotes the formal homogeneous spectrum with respect to a homogeneous ideal I.

II.2.0.5 Remark. We do not assume that R is complete with respect to the I-adic topology. However, replacing R by its completion $R_I^\wedge$ does not change the formal spectrum; there is an isomorphism of locally ringed spaces

$$\mathrm{Spf}(R_I^\wedge, IR_I^\wedge) \xrightarrow{\sim} \mathrm{Spf}(R, I)$$

induced by the completion map $R \to R_I^\wedge$. In the former case, since the ideal for the formal spectrum is clear, we may simply write $\mathrm{Spf}(R_I^\wedge)$.

II.2.0.6 Remark. Although the definition of the formal spectrum of a ring R does not strictly require R to be noetherian, we fix this assumption due to the poorly behaved nature of I-adic completion in the non-noetherian case.

II.2.0.7 Remark. There is a useful way to think about the topological space $|\operatorname{Spf}(R, I)|$. Recall that the support of a (finitely generated) R-module M is

$$\operatorname{Supp}_R(M) = \{\, \mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \neq 0 \,\}.$$

Since

$$S^{-1}(R/I) = 0 \iff S \cap I \neq \varnothing,$$

we see that

$$\operatorname{Supp}_R(R/I) = \{\, \mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p} \,\}.$$

In other words,

$$|\operatorname{Spf}(R, I)| = \operatorname{Supp}_R(R/I).$$

In the same way that the spectrum of a ring globalizes to define schemes, the formal spectrum globalizes to define formal schemes.

II.2.0.8 Definition. Let $(X, \mathcal{O}_X)$ be a locally ringed space. We say that $(X, \mathcal{O}_X)$ is a *noetherian formal scheme* if $|X|$ is quasi-compact and there exists a covering of $|X|$ by open subsets U_α such that each ringed space $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is isomorphic to

$$(|\operatorname{Spf}(R_\alpha, I_\alpha)|, \mathcal{O}_{\operatorname{Spf}(R_\alpha, I_\alpha)})$$

for some noetherian ring R_α and ideal $I_\alpha \subseteq R_\alpha$.

II.2.0.9 Remark. It is clear from the definition that the condition of being a noetherian formal scheme is local: if there exists a covering $\{U_\alpha\}$ of X such that each $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is a noetherian formal scheme, then so is $(X, \mathcal{O}_X)$.

II.2.0.10 Remark. We form the category of noetherian formal schemes as the full subcategory of the category of locally ringed spaces whose objects are noetherian formal schemes. In contrast to some other sources, we do not require the structure sheaf to carry a topological structure (i.e., we work with locally ringed spaces rather than topologically ringed spaces).

Chapter II.3

The Balmer spectrum of a tt-category

We assume the reader has some familiarity with ordinary tt-geometry, for example, the foundational paper [Bal05]. For the benefit of the reader, we briefly recall the basic definitions. We begin by introducing the 'small' and 'big' variants of triangulated categories that will be used; see also [Bar+24, Section 5] for a more thorough discussion.

II.3.0.1 Definition. Let

$$\text{2-Ring} \coloneqq \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})_{\mathrm{rig}}$$

be the ∞-category of commutative algebra objects in $\mathrm{Cat}_\infty^{\mathrm{perf}}$, the ∞-category of small, stable, idempotent-complete ∞-categories and exact functors, equipped with the symmetric monoidal structure described in [BGT13b, Section 3.1]. The subscript rig denotes the full subcategory spanned by those $\mathcal{K}$ for which every object of the underlying symmetric monoidal ∞-category is dualizable. A *2-ring* is then an object $\mathcal{K} \in$ 2-Ring.

We will also need a 'big' variant of this notion.

II.3.0.2 Definition. A *stable homotopy theory* is an object of $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$, i.e., a presentable, symmetric monoidal stable ∞-category $(\mathcal{C}, \otimes, \mathbb{1})$ whose tensor product preserves all colimits separately in each variable. The full subcategory $\mathcal{K} := \mathcal{C}^{\mathrm{d}}$ of dualizable objects is then a 2-Ring in the sense of Definition II.3.0.1 (see [NP24, Lemma 2.5]). A stable homotopy theory is *rigidly-compactly generated* if it is compactly generated and its subcategory of dualizable objects coincides with its subcategory of compact objects.

II.3.0.3 Remark. The homotopy category of a given $\mathcal{K} \in$ 2-Ring is a tensor-triangulated category in the usual sense of [Bal05]. Likewise, if $\mathcal{C}$ is a stable homotopy theory, then its homotopy category is a tensor-triangulated category with small coproducts.

For the definition of the Balmer spectrum $\mathrm{Spc}(\mathcal{K})$ it suffices to work at the level of these underlying triangulated categories. By contrast, our construction of the formal spectrum as a *ringed space* is most naturally formulated after passing to "big" categories, as we will soon describe. We therefore work throughout in the setting of stable ∞-categories: among other advantages, Ind-completion is well behaved in this setting, whereas on the level of triangulated categories the Ind-completion need not remain triangulated; see [Kra23, Remark 5.9] for an example.

II.3.0.4 Remark. By taking Ind-categories, any 2-Ring can be turned into a rigidly-compactly generated stable homotopy theory. Indeed, set $\mathcal{C} := \mathrm{Ind}(\mathcal{K})$. As noted in [NP24], this is presentable, stable, and symmetric monoidal by combining [Lur09c, Theorem 5.5.1.1] and [Lur17a, Corollary 4.8.1.14], and it is rigidly-compactly generated by construction. Moreover, the (restricted) Yoneda embedding $j\colon \mathcal{K} \to \mathcal{C}$ is symmetric monoidal. In fact, if $\mathrm{Pr}^{L,\omega}_{\mathrm{st}}$ denotes the (non-full) symmetric monoidal subcategory of $\mathrm{Pr}^L_{\mathrm{st}}$ spanned by compactly generated stable ∞-categories and colimit-preserving functors that preserve compact objects (equivalently, whose right adjoint preserves colimits), then

Ind-completion defines a symmetric monoidal equivalence

$$\mathrm{Ind}\colon \mathrm{Cat}_{\infty}^{\mathrm{perf}} \xrightarrow{\sim} \mathrm{Pr}_{\mathrm{st}}^{L,\omega},$$

with inverse given by sending $\mathcal{D}$ to $\mathcal{D}^{\omega}$. See [BGT13b, Section 3.1] and [Ben+25, Proposition 2.3]. We will use this equivalence implicitly whenever needed.

Having fixed terminology for both the small and big settings, we now give a standard example that will serve as a running model in what follows.

II.3.0.5 Example. The prototypical example of a 2-Ring is the category Mod_R^c of compact R-modules, where R is a ring spectrum. The associated stable homotopy theory is the category Mod_R of all R-modules. Note that if $R = HA$ is an Eilenberg–MacLane spectrum, these correspond to the derived category of perfect complexes of A-modules and the unbounded derived category of A-modules, respectively.

II.3.0.6 Definition (Balmer [Bal05]). Let $\mathcal{K} \in$ 2-Ring. The *spectrum* of $\mathcal{K}$ is the set

$$\mathrm{Spc}(\mathcal{K}) \coloneqq \{\, \mathcal{P} \mid \mathcal{P} \text{ is a prime thick } \otimes\text{-ideal of } \mathcal{K} \,\}.$$

We topologize this set by declaring a basis of open sets $\{\, U(c) \,\}_{c\in\mathcal{K}}$ defined by

$$U(c) \coloneqq \left\{\, \mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \,\middle|\, c \in \mathcal{P} \,\right\}.$$

II.3.0.7 Definition. The *support* of an object $a \in \mathcal{K}$ is

$$\mathrm{supp}(a) \coloneqq \left\{\, \mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \,\middle|\, a \notin \mathcal{P} \,\right\} = U(a)^{\complement}.$$

The sets $\{\mathrm{supp}(a)\}_{a\in\mathcal{K}}$ form a basis of closed subsets for the topology on $\mathrm{Spc}(\mathcal{K})$.

II.3.0.8 Remark. The spectrum is functorial [Bal05, Proposition 3.6]: given a symmetric monoidal exact functor $F\colon \mathcal{K} \to \mathcal{L}$, there is an induced continuous map

$$\varphi := \mathrm{Spc}(F)\colon\ \mathrm{Spc}(\mathcal{L}) \longrightarrow \mathrm{Spc}(\mathcal{K}), \qquad \mathcal{P} \longmapsto F^{-1}(\mathcal{P}).$$

Moreover, $\varphi^{-1}(\mathrm{supp}_{\mathcal{K}}(a)) = \mathrm{supp}_{\mathcal{L}}(F(a))$ and $\varphi^{-1}(U(c)) = U(F(c))$.

One major reason to study the spectrum is its classification of thick tensor ideals.

II.3.0.9 Theorem (Balmer, [Bal05, Theorem 4.10])**.** *Let $\mathcal{K}$ be a 2-ring. Then support induces an inclusion-preserving bijection:*

$$\Big\{\textit{thick } \otimes\textit{-ideals } \mathcal{J} \subseteq \mathcal{K}\Big\} \leftrightarrows \Big\{\textit{Thomason subsets } Y \subseteq \mathrm{Spc}(\mathcal{K})\Big\}$$

$$\mathcal{J} \longmapsto \mathrm{supp}(\mathcal{J}) := \bigcup_{x\in\mathcal{J}} \mathrm{supp}(x)$$

$$\mathcal{K}_Y := \{\, x \in \mathcal{K} \mid \mathrm{supp}(x) \subseteq Y \,\} \longleftarrow Y.$$

II.3.0.10 Remark. We now describe Balmer's structure sheaf on $\mathrm{Spc}(\mathcal{K})$. Given a thick $\otimes$-ideal $\mathcal{J} \subseteq \mathcal{K}$ as above, we can form the Verdier quotient $\mathcal{K}/\mathcal{J}$. If $U \subset \mathrm{Spc}(\mathcal{K})$ is a quasi-compact open subset with closed complement Z, set

$$\mathcal{K}_Z := \left\{\, a \in \mathcal{K} \;\middle|\; \mathrm{supp}(a) \subseteq Z \,\right\}$$

and define

$$\mathcal{K}(U) := \big(\mathcal{K}/\mathcal{K}_Z\big)^{\natural},$$

where $(-)^{\natural}$ denotes idempotent completion. This is also known as the Karoubi quotient. There is a natural functor $q_U\colon \mathcal{K} \to \mathcal{K}(U)$, and we write $\mathbb{1}_U$ for the image of the unit $\mathbb{1} \in \mathcal{K}$ under q_U.

II.3.0.11 Definition. Let $\mathcal{K} \in$ 2-Ring be an essentially small tt-category. Balmer's structure sheaf $\mathcal{O}_{\mathrm{Spc}(\mathcal{K})}$ on $\mathrm{Spc}(\mathcal{K})$ is the sheafification of the presheaf[1]

$$\widetilde{\mathcal{O}}_{\mathrm{Spc}(\mathcal{K})} \colon \mathrm{Open}(\mathrm{Spc}(\mathcal{K}))^{\mathrm{op}} \longrightarrow \mathrm{grCAlg}^{\heartsuit}$$

given by

$$U \longmapsto \pi_* \operatorname{End}_{\mathcal{K}(U)}(\mathbb{1}_U). \tag{II.3.0.12}$$

II.3.0.13 Remark. Let $\mathcal{K}$ be a 2-ring and $\mathcal{C} := \mathrm{Ind}(\mathcal{K})$ the associated stable homotopy theory. Let $U \subseteq \mathrm{Spc}(\mathcal{K})$ be a quasi-compact open subset with closed complement Z. Consider the localizing subcategory $\mathcal{L} := \mathrm{Loc}\langle \mathcal{K}_Z \rangle \subseteq \mathcal{C}$. Since $\mathcal{K}_Z$ is a thick $\otimes$-ideal in $\mathcal{K}$ and the tensor product in $\mathcal{C}$ preserves colimits separately in each variable, $\mathcal{L}$ is a localizing $\otimes$-ideal in $\mathcal{C}$. Consequently, there is a unique way to equip $\mathcal{C}(U)$ and the localization functor $L \colon \mathcal{C} \to \mathcal{C}(U)$ with symmetric monoidal structures.

We now identify the compact objects of the localization. The restriction of L to dualizable (equivalently, compact) objects, $L|_{\mathcal{K}} \colon \mathcal{K} \to \mathcal{C}(U)$, lands in $\mathcal{C}(U)^c$ (since L preserves compact objects) and vanishes on $\mathcal{K}_Z$. By the universal property of the Verdier quotient, this induces a symmetric monoidal functor

$$\bar{L} \colon \mathcal{K}/\mathcal{K}_Z \longrightarrow \mathcal{C}(U)^c.$$

The Neeman–Thomason localization theorem [Nee92b, Theorem 2.1] & [Cal+26, Theorem A.3.12][2] asserts that $\bar{L}$ is fully faithful and that $\mathcal{C}(U)^c$ identifies with the idempotent completion of the image of $\bar{L}$. Therefore, passing to idempotent completions yields a symmetric monoidal equivalence

$$\mathcal{K}(U) := (\mathcal{K}/\mathcal{K}_Z)^{\natural} \xrightarrow{\sim} (\mathcal{C}(U)^c)^{\natural} \simeq \mathcal{C}(U)^c.$$

[1]Here, and throughout, $\mathrm{grCAlg}^{\heartsuit}$ denotes the category of (discrete) graded commutative rings with the Koszul sign rule.

[2]The cited theorems are essentially the same in different settings. In the setting of triangulated categories in the former and stable ∞-categories in the latter.

II.3.0.14 Remark. If $\mathbb{1}_U$ denotes the tensor unit of the localized category $\mathcal{C}(U)$, then the presheaf from (II.3.0.12) can be rewritten as

$$U \longmapsto \pi_* \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}_U), \tag{II.3.0.15}$$

where in the right-hand side, $\mathbb{1}_U$ is implicitly viewed inside $\mathcal{C}$ via the fully faithful right adjoint to the localization functor L. Although (II.3.0.15) looks slightly less direct than (II.3.0.12), this ambient formulation in $\mathcal{C}$ will be crucial for the construction of the formal Balmer spectrum, where we will encounter localizations that do not preserve compact objects, necessitating the use of the large category $\mathcal{C}$.

II.3.0.16 Remark. In [Aok+25, Theorem D and Theorem F], it is shown that for any 2-Ring $\mathcal{K}$ with associated stable homotopy theory $\mathcal{C} := \operatorname{Ind}(\mathcal{K})$, the functor sending a quasi-compact open $U \subseteq \operatorname{Spc}(\mathcal{K})$ to the ∞-category $\mathcal{C}(U)$ refines to a sheaf

$$\mathcal{O}_{\mathcal{K}} \colon \operatorname{Open}(\operatorname{Spc}(\mathcal{K}))^{\mathrm{op}} \longrightarrow \operatorname{CAlg}(\operatorname{Pr}^L_{\mathrm{st}}).$$

II.3.0.17 Remark. It follows from [Lur17a, Theorem 4.8.5.11] and [Lur17a, Theorem 4.8.5.16] that the functor

$$\operatorname{Alg}(\mathcal{S}p) \longrightarrow \operatorname{Pr}^L_{\mathrm{st}},$$

sending an associative algebra A in spectra to the stable ∞-category $\operatorname{LMod}_A(\mathcal{S}p)$ of left A-module spectra, is symmetric monoidal and admits a right adjoint. In particular, by Dunn additivity and passing to commutative algebra objects, we obtain a right adjoint functor

$$\operatorname{End}(\mathbb{1}_{(-)}) \colon \operatorname{CAlg}(\operatorname{Pr}^L_{\mathrm{st}}) \longrightarrow \operatorname{CAlg}(\mathcal{S}p), \tag{II.3.0.18}$$

sending a stable homotopy theory $\mathcal{C}$ to the commutative algebra in spectra given by the endomorphism spectrum $\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ (see also [Aok+25, Construction 4.29]).

If we compose $\mathcal{O}_{\mathcal{K}}$ with $\mathrm{End}(\mathbb{1}_{(-)})$ we obtain a sheaf

$$\mathcal{O}^{\mathbb{1}}_{\mathcal{K}} \colon \mathrm{Open}(\mathrm{Spc}(\mathcal{K}))^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathcal{S}p)$$

of commutative algebras in spectra. This sheaf is given on a quasi-compact open $U \subseteq \mathrm{Spc}(\mathcal{K})$ by $\mathrm{Hom}_{\mathcal{C}(U)}(\mathbb{1}_U, \mathbb{1}_U) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}_U)$ in the notation of Remark II.3.0.13.

In particular, Balmer's structure sheaf of graded rings $\mathcal{O}_{\mathrm{Spc}(\mathcal{K})}$ is the induced sheaf obtained from the functor $\pi_* \colon \mathrm{CAlg}(\mathcal{S}p) \to \mathrm{grCAlg}^{\heartsuit}$.

II.3.0.19 Remark (Functoriality). The functoriality of Remark II.3.0.8 is compatible with this ringed space structure. That is, given a tt-functor $F \colon \mathcal{K} \to \mathcal{L}$, there is a morphism of ringed spaces

$$\mathrm{Spc}(F) \coloneqq \varphi \colon (|\mathrm{Spc}(\mathcal{L})|, \mathcal{O}_{\mathrm{Spc}(\mathcal{L})}) \to (|\mathrm{Spc}(\mathcal{K})|, \mathcal{O}_{\mathrm{Spc}(\mathcal{K})}).$$

For details, see [BKS07, Lemma 7.2].

II.3.0.20 Example. The most fundamental example comes from work of Hopkins and Neeman in the affine case, as extended to schemes by Thomason [Hop87; Nee92a; Tho97]. Let X be a quasi-compact and quasi-separated scheme. There is an isomorphism of locally ringed spaces (and hence schemes)

$$f \colon X \xrightarrow{\sim} \mathrm{Spc}(\mathrm{D}_{\mathrm{qc}}(X)^c)$$

given by

$$f(x) = \left\{ a \in \mathrm{D}_{\mathrm{qc}}(X)^c \;\middle|\; a_x \simeq 0 \text{ in } \mathrm{D}_{qc}(\mathcal{O}_{X,x})^c \right\} \quad \text{for all } x \in X.$$

Moreover, under this equivalence, the homological support[1] $\mathrm{supph}(a) \subseteq X$ of a perfect complex $a \in \mathrm{D}_{\mathrm{qc}}(X)^c$ corresponds to the closed subset $\mathrm{supp}(a) \subseteq \mathrm{Spc}(\mathrm{D}_{\mathrm{qc}}(X)^c)$.

For the translation to tt-geometry in this form, see [Bal05, Corollary 5.6 and Theorem 6.3] and [BKS07, Theorem 8.5].

[1] That is, the support of the total homology of a.

II.3.0.21 Remark. When $X = \operatorname{Spec}(R)$ is affine, there is a simple description of the inverse to the above isomorphism, given by the following general construction and theorem of Balmer.

II.3.0.22 Theorem (Balmer)**.** *Let $\mathcal{K} \in$ 2-Ring be an essentially small tt-category. Then there is a natural continuous, inclusion-reversing map*

$$\rho\colon \operatorname{Spc}(\mathcal{K}) \to \operatorname{Spec}^h(\pi_* \operatorname{End}_{\mathcal{K}}(\mathbb{1}))$$

of locally ringed spaces, given by

$$\rho(\mathcal{P}) = \left\{ f \;\middle|\; \operatorname{cone}(f) \notin \mathcal{P} \right\}.$$

II.3.0.23 Remark. It is not always the case that $\operatorname{Spc}(\mathcal{K})$ is a scheme; for instance, taking $\mathcal{K} = \mathrm{SH}^c$ gives a counterexample. See [Bal10, Proposition 9.7].

Chapter II.4

Completions of tt-categories

The tensor-triangular analog of I-adic completion is completion with respect to a closed subset of the Balmer spectrum. In this short section we recall the notion, and review the most basic example of the derived category of a commutative ring. Nothing in this section is new; the ideas date back to at least Greenlees [Gre01], and we follow the recent exposition in [BS25].

II.4.0.1 Definition. Let $\mathcal{C}$ be a rigidly compactly generated stable homotopy theory, and let $Y \subseteq \mathrm{Spc}(\mathcal{C}^d)$ be a Thomason subset. We set

$$\mathcal{C}_Y^d \coloneqq \left\{ x \in \mathcal{C}^d \,\middle|\, \mathrm{supp}(x) \subseteq Y \right\} \quad \text{and} \quad \mathcal{C}_Y \coloneqq \mathrm{Loc}\langle \mathcal{C}_Y^d \rangle.$$

II.4.0.2 Remark. The inclusion $\mathcal{C}_Y \hookrightarrow \mathcal{C}$ admits a right adjoint given by tensoring with the Balmer–Favi idempotent e_Y [BF11]. Our primary interest, however, lies in the double right orthogonal to $\mathcal{C}_Y$. Recall that if $\mathcal{E} \subseteq \mathcal{C}$ is any class of objects, we write

$$\mathcal{E}^{\perp} \coloneqq \left\{ c \in \mathcal{C} \,\middle|\, \mathrm{Hom}_{\mathcal{C}}(s, c) = 0 \text{ for all } s \in \mathcal{E} \right\}.$$

Note that this is closed under suspension in $\mathcal{C}$.

II.4.0.3 Definition. The *completion* of $\mathcal{C}$ at $Y \subseteq \mathrm{Spc}(\mathcal{C}^d)$ is the full subcategory

$$\widehat{\mathcal{C}}_Y \coloneqq (\mathcal{C}_Y)^{\perp\perp}.$$

Objects of $\widehat{\mathcal{C}}_Y$ are called *Y-complete.*

II.4.0.4 Remark. The inclusion $\widehat{\mathcal{C}}_Y \hookrightarrow \mathcal{C}$ has a symmetric monoidal left adjoint, given by $\widehat{(-)}_Y \coloneqq \mathsf{hom}(e_Y, -)$. The category $\widehat{\mathcal{C}}_Y$ is a tensor-triangulated category under the localized tensor product. It is not, however, rigidly-compactly generated, except in trivial cases (see [BS24, Remark 2.12]). Note that via the inclusion we can also consider $\widehat{(-)}_Y$ as an endofunctor on $\mathcal{C}$.

II.4.0.5 Remark. We are often interested in the case where $Y = \mathrm{supp}(A)$ for some $A \in \mathcal{C}^d$. In this case, the completion functor $\widehat{(-)}_{\mathrm{supp}(A)}$ coincides with Bousfield localization with respect to A (see, for example, [BHV18, Proposition 2.34]).

II.4.0.6 Example. The most basic example is as follows. Let R be a noetherian ring, $I \subseteq R$ an ideal, and consider the closed subset $Y = V(I) \subseteq \mathrm{Spec}(R) \cong \mathrm{Spc}(\mathrm{D}(R)^c)$ (see Example II.3.0.20). Then the functor

$$\widehat{(-)}_I \colon \mathrm{D}(R) \longrightarrow \widehat{\mathrm{D}(R)}_I \coloneqq \widehat{\mathrm{D}(R)}_{V(I)}$$

is the derived I-adic completion functor; see, for instance, [GM92; DG02]. Note that even if M is a discrete R-module, the derived completion $M_I^\wedge$ need not agree with the classical I-adic completion. For example, with $R = \mathbb{Z}$, $I = (p)$, and $M = \bigoplus_{n \geq 1} \mathbb{Z}/p^n$, the two notions differ [MP12, Example 10.1.16].

Chapter II.5

The formal spectrum of a tt-category

We now turn to the definition of the formal spectrum in the tensor-triangular setting.

II.5.0.1 Definition. Let $\mathcal{K} \in$ 2-Ring be a 2-ring and let $\mathcal{J} = \mathcal{K}_Y$ be the thick $\otimes$-ideal in $\mathcal{K}$ corresponding to a Thomason subset $Y \subseteq \operatorname{Spc}(\mathcal{K})$ (note that every thick $\otimes$-ideal is of this form by Theorem II.3.0.9). Write $\mathcal{C}$ for the stable homotopy theory associated to $\mathcal{K}$ (Remark II.3.0.4). The *formal spectrum* of $\mathcal{K}$ (with respect to Y), denoted $\operatorname{Spf}(\mathcal{K}, Y)$, is the ringed space

$$(|\operatorname{Spf}(\mathcal{K}, Y)|, \mathcal{O}_{\operatorname{Spf}(\mathcal{K}, Y)})$$

defined as follows:

(a) The underlying topological space is

$$|\operatorname{Spf}(\mathcal{K}, Y)| \coloneqq Y \subseteq \operatorname{Spc}(\mathcal{K})$$

equipped with the subspace topology. A basis of open sets is given by

$$\overline{U}(c) \coloneqq U(c) \cap Y = \left\{ \mathcal{P} \in Y \,\middle|\, c \in \mathcal{P} \right\}.$$

(b) The structure sheaf is given on basic opens $\overline{U}(c)$ by the sheafification (in $\mathrm{grCAlg}^{\heartsuit}$) of the presheaf

$$\overline{U}(c) \longmapsto \pi_* \operatorname{Hom}_{\mathcal{C}}\Big(\mathbb{1}, (\widehat{\mathbb{1}[c^{-1}]})_Y\Big),$$

where $\mathbb{1}[c^{-1}]$ denotes the tensor unit of the localization $\mathcal{C}[c^{-1}] = \mathcal{C}(U(c))$, viewed as an object of $\mathcal{C}$ via the right adjoint to the localization functor, and $\widehat{(-)}_Y$ denotes Y-completion as introduced in Chapter II.4.

II.5.0.2 Question. Is $\mathrm{Spf}(\mathcal{K}, Y)$ always a *locally* ringed space?

II.5.0.3 Remark. Recall that $\widehat{(-)}_Y \simeq \mathsf{hom}(e_Y, -)$. Therefore, the presheaf can equivalently be written as

$$\overline{U}(c) \longmapsto \pi_* \operatorname{Hom}_{\mathcal{C}}\big(e_Y, \mathbb{1}[c^{-1}]\big).$$

II.5.0.4 Remark. Recall from Remark II.3.0.14 there is a sheaf

$$\mathcal{O}_{\mathcal{K}}^{\mathbb{1}} \colon \operatorname{Open}(\operatorname{Spc}(\mathcal{K}))^{\mathrm{op}} \longrightarrow \operatorname{CAlg}(\mathcal{S}p)$$

of commutative algebras in spectra. On a quasi-compact open U the value of the sheaf is given by $\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}_U)$. If $Y \subseteq \operatorname{Spc}(\mathcal{K})$ is a Thomason subset and e_Y is the associated Balmer-Favi idempotent, then the association sending a quasi-compact open $U \subseteq \operatorname{Spc}(\mathcal{K})$ to $\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, (\widehat{\mathbb{1}_U})_Y)$ defines a functor

$$\mathcal{O}_{\mathcal{K}_Y} \colon \operatorname{Open}(\operatorname{Spc}(\mathcal{K}))^{\mathrm{op}} \longrightarrow \operatorname{CAlg}(\mathcal{S}p).$$

We claim that $\mathcal{O}_{\mathcal{K}_Y}$ is a sheaf. By combining [Lur17a, Theorem 4.8.5.11], [Lur17a, Theorem 4.8.5.16] and [Aok+25, Corollary 5.41] we see that the functor

$$\operatorname{Open}(\operatorname{Spc}(\mathcal{K}))^{\mathrm{op}} \longrightarrow \operatorname{CAlg}(\mathcal{C})$$

sending a quasi-compact open $U \subseteq \mathrm{Spc}(\mathcal{K})$ to $\mathbb{1}_U$ is a sheaf. In particular, it suffices to see that the functor sending U to $\widehat{(\mathbb{1}_U)}_Y$ is a sheaf. This following from [Lur09c, Theorem 7.3.5.2], since $\widehat{(-)}_Y : \mathcal{C} \to \mathcal{C}$ preserves limits and $\mathrm{Spc}(\mathcal{K})$ is spectral.

We claim this sheaf is extended from its restriction to $|\operatorname{Spf}(\mathcal{K}, Y)| = Y$. It suffices to see that for any quasi-compact open $U \subseteq \mathrm{Spc}(\mathcal{K})$ such that $U \cap Y = \varnothing$ the commutative algebra in spectra of sections $\mathcal{O}_{\mathcal{K}_Y}(U) \simeq 0$ is trivial. This follows from the calculus of idempotents coming from Thomason subsets of $\mathrm{Spc}(\mathcal{K})$ presented in [BF11, Theorem 5.18]. Indeed, let Z be the closed complement of U, so $\mathbb{1}_U \simeq f_Z$. If $U \cap Y = \varnothing$ then $Y \subseteq Z$, hence $e_Y \otimes e_Z \simeq e_Y$. Tensoring the triangle $e_Z \to \mathbb{1} \to f_Z$ with e_Y shows $e_Y \otimes f_Z \simeq 0$, so $\mathrm{Hom}_{\mathcal{C}}(e_Y, f_Z) \simeq \mathrm{Hom}_{\mathcal{C}}(e_Y \otimes f_Z, f_Z) \simeq 0$, i.e. $\mathcal{O}_{\mathcal{K}_Y}(U) \simeq 0$.

We deduce, like in Remark II.3.0.17, that the structure sheaf of graded commutative rings defined in Definition II.5.0.1 is induced from $\mathcal{O}_{\mathcal{K}_Y}|_Y$ via $\pi_* \colon \mathrm{CAlg}(\mathcal{S}p) \to \mathrm{grCAlg}^{\heartsuit}$.

II.5.0.5 Remark. Unlike in Remark II.3.0.16, we are currently unable to construct a sheaf valued in $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$. We hope to pursue this further in future work.

II.5.0.6 Remark. Taking $c = 0$ (which lies in every prime), the value on $\overline{U}(0) = Y$ is

$$Y \longmapsto \pi_* \mathrm{Hom}_{\mathcal{C}}\big(\mathbb{1}, \widehat{\mathbb{1}}_Y\big) \cong \pi_* \mathrm{Hom}_{\mathcal{C}}\big(e_Y, \mathbb{1}\big).$$

II.5.0.7 Example. Taking $\mathcal{J} = \mathrm{thick}_{\otimes}\langle \mathbb{1} \rangle$ (equivalently $Y = \mathrm{Spc}(\mathcal{K})$), the construction reduces to the usual Balmer spectrum:

$$\operatorname{Spf}(\mathcal{K}, \mathrm{Spc}(\mathcal{K})) \cong \mathrm{Spc}(\mathcal{K})$$

as ringed spaces (compare with (II.3.0.15)).

II.5.0.8 Remark. There is a subtle difference from the algebraic situation: for rings, one typically forms formal spectra only along *closed* subsets of $\mathrm{Spec}(R)$, whereas here we allow arbitrary Thomason subsets $Y \subseteq \mathrm{Spc}(\mathcal{K})$.

The formal spectrum is functorial in the obvious way:

II.5.0.9 Lemma. *Let $\mathcal{K}, \mathcal{L}$ be 2-rings and let $Y \subseteq \mathrm{Spc}(\mathcal{K})$ and $Y' \subseteq \mathrm{Spc}(\mathcal{L})$ be Thomason subsets. Suppose $F\colon \mathcal{K} \to \mathcal{L}$ is an exact symmetric monoidal functor and write $\varphi := \mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ for the induced map. If $\varphi(Y') \subseteq Y$, then there is a morphism of ringed spaces*

$$(\varphi, \varphi^{\#})\colon \mathrm{Spf}(\mathcal{L}, Y') \longrightarrow \mathrm{Spf}(\mathcal{K}, Y).$$

Proof. On underlying spaces this is the restriction $\varphi|_{Y'}\colon Y' \to Y$. For the sheaf map, we set $\mathcal{C} := \mathrm{Ind}(\mathcal{K})$ and $\mathcal{D} := \mathrm{Ind}(\mathcal{L})$; the functor F extends to a colimit-preserving symmetric monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$.

Let $\overline{U}(c) := U(c) \cap Y$ be a basic open of $\mathrm{Spf}(\mathcal{K}, Y)$, with $c \in \mathcal{K}$ compact. Using Remark II.5.0.3 we have

$$\mathcal{O}_{\mathrm{Spf}(\mathcal{K},Y)}(\overline{U}(c)) \cong \pi_* \mathrm{Hom}_{\mathcal{C}}(e_Y, \mathbb{1}[c^{-1}]).$$

Define on the basis the composite

$$\begin{aligned}\pi_* \mathrm{Hom}_{\mathcal{C}}(e_Y, \mathbb{1}[c^{-1}]) &\longrightarrow \pi_* \mathrm{Hom}_{\mathcal{D}}(F(e_Y), F(\mathbb{1}[c^{-1}]))\\ &\xrightarrow{\sim} \pi_* \mathrm{Hom}_{\mathcal{D}}(e_{\varphi^{-1}(Y)}, \mathbb{1}[F(c)^{-1}])\\ &\xrightarrow{\mathrm{res}} \pi_* \mathrm{Hom}_{\mathcal{D}}(e_{Y'}, \mathbb{1}[F(c)^{-1}]),\end{aligned}$$

where $F(e_Y) \simeq e_{\varphi^{-1}(Y)}$ and $F(\mathbb{1}[c^{-1}]) \simeq \mathbb{1}[F(c)^{-1}]$ by the functoriality of idempotents (see [BS17, Proposition 5.11]), and the last map is induced by the morphism of idempotents $e_{Y'} \to e_{\varphi^{-1}(Y)}$ coming from $Y' \subseteq \varphi^{-1}(Y)$. Note that the target is precisely $\mathcal{O}_{\mathrm{Spf}(\mathcal{L},Y')}(\overline{U}(F(c)))$, where $\overline{U}(F(c)) := U(F(c)) \cap Y'$ and $\overline{U}(F(c)) = \varphi^{-1}(\overline{U}(c))$ as open sets of Y'. The constructions are all natural in c and compatible with restriction maps, and hence define a morphism

$$\varphi^{\#}\colon \mathcal{O}_{\mathrm{Spf}(\mathcal{K},Y)} \longrightarrow \varphi_* \mathcal{O}_{\mathrm{Spf}(\mathcal{L},Y')}.$$

We conclude that, $(\varphi, \varphi^{\#})$ is a morphism of ringed spaces $\mathrm{Spf}(\mathcal{L}, Y') \to \mathrm{Spf}(\mathcal{K}, Y)$. $\square$

II.5.0.10 Example. As a special case of Lemma II.5.0.9, take $F = \mathrm{id}$ and $Y = \mathrm{Spc}(\mathcal{K})$. Together with Example II.5.0.7, this shows that for any Thomason subset $Y' \subseteq \mathrm{Spc}(\mathcal{K})$, there is a canonical morphism of ringed spaces

$$\mathrm{Spf}(\mathcal{K}, Y') \longrightarrow \mathrm{Spc}(\mathcal{K}),$$

which realizes the inclusion $Y' \hookrightarrow \mathrm{Spc}(\mathcal{K})$ on underlying spaces. Here the morphism of sheaves is induced by the map $e_{Y'} \to \mathbb{1}$.

II.5.0.11 Notation. Let $\mathcal{K}$ be a 2-ring, and write $\mathcal{C}$ for the associated stable homotopy theory. For a Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{K})$, consider the functor

$$\widehat{(-)}_Y \colon \mathcal{C} \to \widehat{\mathcal{C}}_Y.$$

This is symmetric monoidal, and so restricts to a functor of 2-rings

$$\widehat{(-)}_Y \colon \mathcal{C}^{\mathrm{d}} = \mathcal{K} \to \widehat{\mathcal{C}}^{\mathrm{d}}_Y. \tag{II.5.0.12}$$

By Lemma II.5.0.9 we therefore get a morphism of ringed spaces

$$(\varphi, \varphi^{\#}) \colon \mathrm{Spf}(\widehat{\mathcal{C}}^{\mathrm{d}}_Y, \varphi^{-1}(Y)) \longrightarrow \mathrm{Spf}(\mathcal{K}, Y). \tag{II.5.0.13}$$

The crucial ingredient for the following result is due to Balmer and Sanders [BS25]. Note that this is the analog of Remark II.2.0.5.

II.5.0.14 Theorem. *Let $\mathcal{K}$ be a 2-ring and let $Y \subseteq \mathrm{Spc}(\mathcal{K})$ be a Thomason subset. Then the canonical morphism*

$$(\varphi, \varphi^{\#}) \colon \mathrm{Spf}(\widehat{\mathcal{C}}^{\mathrm{d}}_Y, \varphi^{-1}(Y)) \longrightarrow \mathrm{Spf}(\mathcal{K}, Y)$$

is an isomorphism of ringed spaces.

Proof. On underlying topological spaces, φ is a homeomorphism by [BS25, Theorem 4.6]. It remains to show that $\varphi^{\#}$ is an isomorphism of sheaves. Since both structure sheaves are defined as sheafifications of presheaves on the basis $\{\overline{U}(c)\}$, it suffices to check that $\varphi^{\#}$ is an isomorphism on each basic open set.

Fix a basic open $\overline{U}(c)$ and set $X := \mathbb{1}[c^{-1}] \in \mathcal{C}$. Using Remark II.5.0.3, the sections of $\mathcal{O}_{\mathrm{Spf}(\mathcal{K},Y)}$ over $\overline{U}(c)$ is $\pi_* \operatorname{Hom}_{\mathcal{C}}(e_Y, X)$. On the other hand, in $\widehat{\mathcal{C}}_Y$ we have $L = \widehat{(-)}_Y \simeq \mathsf{hom}(e_Y, -)$, and hence the corresponding ring of sections is $\pi_* \operatorname{Hom}_{\widehat{\mathcal{C}}_Y}(L(e_Y), L(X))$. The map

$$\varphi^{\#}_{\overline{U}(c)} \colon \pi_* \operatorname{Hom}_{\mathcal{C}}(e_Y, X) \longrightarrow \pi_* \operatorname{Hom}_{\widehat{\mathcal{C}}_Y}(L(e_Y), L(X))$$

is induced by the symmetric monoidal functor L.

We claim that the underlying map of mapping spectra

$$\operatorname{Hom}_{\mathcal{C}}(e_Y, X) \longrightarrow \operatorname{Hom}_{\widehat{\mathcal{C}}_Y}(L(e_Y), L(X))$$

is an equivalence. Indeed, by the adjunction $L \dashv \iota$ and the identification $\iota L(-) \simeq \mathsf{hom}(e_Y, -)$, there are natural equivalences

$$\begin{aligned}\operatorname{Hom}_{\widehat{\mathcal{C}}_Y}(L(e_Y), L(X)) &\simeq \operatorname{Hom}_{\mathcal{C}}(e_Y, \iota L(X))\\ &\simeq \operatorname{Hom}_{\mathcal{C}}(e_Y, \mathsf{hom}(e_Y, X))\\ &\simeq \operatorname{Hom}_{\mathcal{C}}(e_Y \otimes e_Y, X)\\ &\simeq \operatorname{Hom}_{\mathcal{C}}(e_Y, X),\end{aligned}$$

where the third equivalence is the tensor–hom adjunction in $\mathcal{C}$ and the last uses the idempotence $e_Y \otimes e_Y \simeq e_Y$. Under these identifications, the map induced by L corresponds to the composite above, hence is an equivalence. Applying π_* shows that $\varphi^{\#}_{\overline{U}(c)}$ is an isomorphism for every basic open $\overline{U}(c)$, and therefore $\varphi^{\#}$ is an isomorphism of sheaves. $\square$

II.5.0.15 Remark. Because of this theorem, we will usually write $\mathrm{Spf}(\widehat{\mathcal{C}}^{\mathrm{d}}_Y)$ instead of the more cumbersome $\mathrm{Spf}(\widehat{\mathcal{C}}^{\mathrm{d}}_Y, \varphi^{-1}(Y))$.

We finish this section with the following locality theorem.

II.5.0.16 Theorem. *Let $\mathcal{K}$ be a 2-ring and $Y \subseteq \mathrm{Spc}(\mathcal{K})$ a Thomason subset. Let $U \subseteq \mathrm{Spc}(\mathcal{K})$ be a quasi-compact open subset, and let $q_U \colon \mathcal{K} \to \mathcal{K}(U)$ be the functor from Remark II.3.0.10. Set $Y' := Y \cap U \subseteq \mathrm{Spc}(\mathcal{K}(U))$. Then the restriction of $\mathrm{Spf}(\mathcal{K}, Y)$ to Y' identifies with $\mathrm{Spf}(\mathcal{K}(U), Y')$ as a ringed space:*

$$\mathrm{Spf}(\mathcal{K}(U), Y') \xrightarrow{\cong} \mathrm{Spf}(\mathcal{K}, Y)\big|_{Y'}.$$

Proof. Since U is quasi-compact, there exists $d \in \mathcal{K}$ such that $U = U(d)$ [Bal05, Proposition 2.14]. Consequently, the localization $\mathcal{K}(U)$ is identified with the idempotent completion of the Verdier quotient, $(\mathcal{K}[d^{-1}])^\natural$.

By functoriality (Lemma II.5.0.9) applied to q_U, we obtain a morphism

$$(\varphi, \varphi^\#) \colon \mathrm{Spf}(\mathcal{K}(U), Y') \longrightarrow \mathrm{Spf}(\mathcal{K}, Y).$$

On underlying topological spaces, φ is the inclusion $Y' \hookrightarrow Y$. Since the image of φ lies in the open set Y', this morphism factors through the restriction $\mathrm{Spf}(\mathcal{K}, Y)|_{Y'}$. To prove that this factorization is an isomorphism, it remains to show that the induced map on structure sheaves is an isomorphism. It suffices to verify this on a basis of the open sets of Y'.

Let $\overline{V} \subseteq Y'$ be a basic open set. We can write $\overline{V} = U(c) \cap Y'$ for some $c \in \mathcal{K}$. By replacing c with $c \oplus d$ (which satisfies $U(c \oplus d) = U(c) \cap U(d)$), we may assume without loss of generality that $U(c) \subseteq U(d) = U$.

Let $\mathcal{C} := \mathrm{Ind}(\mathcal{K})$ and let $L \colon \mathcal{C} \to \mathcal{C}(U)$ be the localization functor, with fully faithful right adjoint ι. We compare the sections over $\overline{V}$:

(a) On $\mathrm{Spf}(\mathcal{K}, Y)$: Since $U(c) \cap Y = \overline{V}$ (as $U(c) \subseteq U$), the section is

$$\pi_* \operatorname{Hom}_{\mathcal{C}}(e_Y, \mathbb{1}_{\mathcal{C}}[c^{-1}]).$$

(b) On $\mathrm{Spf}(\mathcal{K}(U), Y')$: The section is

$$\pi_* \operatorname{Hom}_{\mathcal{C}(U)}(e_{Y'}, \mathbb{1}_{\mathcal{C}(U)}[c^{-1}]).$$

Since $U(c) \subseteq U(d)$, the object $\mathbb{1}_{\mathcal{C}}[c^{-1}]$ is U-local. Thus, the canonical map $\mathbb{1}_{\mathcal{C}}[c^{-1}] \to \iota(\mathbb{1}_{\mathcal{C}(U)}[c^{-1}])$ is an equivalence. Using the adjunction $L \dashv \iota$ and the fact that $L(e_Y) \simeq e_{Y \cap U} = e_{Y'}$ (see [BS17, Proposition 5.11]), we obtain natural isomorphisms:

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}(e_Y, \mathbb{1}_{\mathcal{C}}[c^{-1}]) &\cong \operatorname{Hom}_{\mathcal{C}(U)}(L(e_Y), \mathbb{1}_{\mathcal{C}(U)}[c^{-1}]) \\ &\cong \operatorname{Hom}_{\mathcal{C}(U)}(e_{Y'}, \mathbb{1}_{\mathcal{C}(U)}[c^{-1}]). \end{aligned}$$

This shows the map of sheaves is an isomorphism on basic open sets, completing the proof. $\square$

Chapter II.6

The formal Hopkins–Neeman theorem

Let R be a noetherian ring and let $I \subseteq R$ be an ideal. In Example II.4.0.6, we introduced the derived category of I-complete R-modules, $\widehat{\mathrm{D}(R)}_I$, as the completion of $\mathrm{D}(R)$ along the closed subset $V(I) \subseteq \mathrm{Spc}(\mathrm{D}(R)^c)$. In this section, we compute the formal spectrum $\mathrm{Spf}(\mathrm{D}(R)^c, V(I))$ and identify it with the classical affine formal scheme $\mathrm{Spf}(R_I^\wedge)$.

II.6.0.1 Remark. We first recall some properties of the derived I-completion functor $\widehat{(-)}_I \colon \mathrm{D}(R) \to \widehat{\mathrm{D}(R)}_I$ from Example II.4.0.6. For a complex $M \in \mathrm{D}(R)$, the object $\widehat{M}_I$ is the *derived* I-adic completion of M. Even if M is a discrete module, $\widehat{M}_I$ may differ from the classical I-adic completion. In general, [GM92, Proposition 1.1] provides a short exact sequence for any discrete module M:

$$0 \to \varprojlim_k{}^1 \mathrm{Tor}_{i+1}^R(R/I^k, M) \to \pi_i \widehat{M}_I \to \varprojlim_k \mathrm{Tor}_i^R(R/I^k, M) \to 0.$$

This implies the following useful lemma:

II.6.0.2 Lemma. *If M is a flat discrete R-module, then the natural map $\widehat{M}_I \to M_I^\wedge$ is an isomorphism, where $M_I^\wedge$ denotes the ordinary I-adic completion.*

Our version of the Hopkins–Neeman theorem is the following:

II.6.0.3 Theorem. *Let R be a commutative noetherian ring and let $I \subseteq R$ be an ideal. Then there is an isomorphism of locally ringed spaces (and hence, formal schemes)*

$$\rho\colon \operatorname{Spf}\big(\mathrm{D}(R)^c, V(I)\big) \xrightarrow{\sim} \operatorname{Spf}(R_I^\wedge)$$

induced by the comparison map of Theorem II.3.0.22. Consequently, we have an isomorphism

$$\operatorname{Spf}\big(\widehat{\mathrm{D}(R)}_I^{\mathrm{d}}\big) \xrightarrow{\sim} \operatorname{Spf}(R_I^\wedge).$$

Proof. By Remark II.2.0.5 and Theorem II.5.0.14, it suffices to establish the isomorphism

$$\rho\colon \operatorname{Spf}\big(\mathrm{D}(R)^c, V(I)\big) \xrightarrow{\sim} \operatorname{Spf}(R, I).$$

In this context, the stable homotopy theory associated to $\mathrm{D}(R)^c$ is $\mathrm{D}(R)$.

On the level of topological spaces, the result follows from Example II.3.0.20: the isomorphism $\operatorname{Spc}(\operatorname{Perf}(R)) \cong \operatorname{Spec}(R)$ identifies the tensor-triangular support with the classical support of R-modules, mapping $V(I)$ homeomorphically to itself.

It remains to identify the structure sheaves. We check this on the basis of open sets $U = \overline{D}(s) = D(s) \cap V(I)$. Recall from [Bal10, Theorem 5.3] that under ρ, the preimage of $D(s)$ is the open set $U(\operatorname{cone}(s))$. Thus,

$$\rho^{-1}\big(\overline{D}(s)\big) = U(\operatorname{cone}(s)) \cap V(I) = \overline{U}(\operatorname{cone}(s)).$$

The ring of sections of the structure sheaf of $\mathrm{Spf}(R, I)$ over this open set is simply

$$\mathcal{O}_{\mathrm{Spf}(R,I)}\big(\overline{D}(s)\big) = R[1/s]^{\wedge}_{I}.$$

On the tensor-triangular side, since R is the tensor unit of $\mathrm{D}(R)$, the ring of sections is given by

$$\begin{aligned}\mathcal{O}_{\mathrm{Spf}(\mathrm{D}(R)^c, V(I))}\big(\overline{U}(\mathrm{cone}(s))\big) &= \pi_* \operatorname{Hom}_{\mathrm{D}(R)}\big(R, (\widehat{\mathbb{1}[\mathrm{cone}(s)^{-1}]})_I\big) \\ &\cong \pi_*\big((\widehat{\mathbb{1}[\mathrm{cone}(s)^{-1}]})_I\big).\end{aligned}$$

Here, $\mathbb{1}[\mathrm{cone}(s)^{-1}]$ denotes the tensor unit in the localization of $\mathrm{D}(R)$ at the open set $U(\mathrm{cone}(s))$. This localization corresponds to the Verdier quotient by the subcategory generated by $\mathrm{cone}(s)$. As shown in [HPS97, Theorem 3.3.7], this quotient identifies with $\mathrm{D}(R[1/s])$ as tensor-triangulated categories. Under this equivalence, the localized unit $\mathbb{1}[\mathrm{cone}(s)^{-1}]$ corresponds to the ring $R[1/s]$. Therefore, we have an equivalence of completions

$$(\widehat{\mathbb{1}[\mathrm{cone}(s)^{-1}]})_I \simeq \widehat{R[1/s]}_I.$$

Since $R[1/s]$ is a flat R-module, its derived completion coincides with its classical completion by Lemma II.6.0.2. Combining these results yields a natural isomorphism

$$\mathcal{O}_{\mathrm{Spf}(\mathrm{D}(R)^c, V(I))}\big(\overline{U}(\mathrm{cone}(s))\big) \cong R[1/s]^{\wedge}_{I} \cong \mathcal{O}_{\mathrm{Spf}(R,I)}\big(\overline{D}(s)\big).$$

These local isomorphisms are compatible with restriction, so they glue to an isomorphism of structure sheaves. Thus, ρ is an isomorphism of ringed spaces. Finally, since $\mathrm{Spf}(R, I)$ is a locally ringed space, the isomorphism means that $\mathrm{Spf}(\mathrm{D}(R)^c, V(I))$ is as well, and by definition, this means it is an isomorphism of formal schemes. □

II.6.0.4 Remark. This result will also follow from results in Chapter II.8, in particular Proposition II.8.0.6. However we believe it is still instructive to do this special case.

Chapter II.7

Globalizing to schemes

We can now globalize the formal Hopkins–Neeman theorem (Theorem II.6.0.3) to arbitrary noetherian schemes. We begin by recalling the notion of completion in the setting of schemes.

II.7.0.1 Definition. Let X be a noetherian scheme, and let $Y \subseteq |X|$ be a closed subset. An *ideal of definition* for Y is a quasi-coherent ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ of finite type such that $|V(\mathcal{J})| = Y$.

II.7.0.2 Definition. Let X be a quasi-compact quasi-separated noetherian scheme, and $Y \subseteq |X|$ a closed subset with ideal of definition $\mathcal{J}$. For $n \geq 0$, set

$$X_n \coloneqq (V(\mathcal{J}^{n+1}),\ \mathcal{O}_X/\mathcal{J}^{n+1}).$$

The colimit in the category of locally ringed spaces

$$\widehat{X}_Y \coloneqq \varinjlim_n X_n$$

is called the *formal completion* of X along Y.

II.7.0.3 Remark. Explicitly, $\widehat{X}_Y$ is the locally ringed space whose underlying topological space is Y, equipped with the structure sheaf

$$\mathcal{O}_{\widehat{X}_Y} \coloneqq \varprojlim_n \mathcal{O}_X/\mathcal{J}^{n+1}.$$

By [GD71, Proposition 10.8.5], $\widehat{X}_Y$ is a formal scheme.

II.7.0.4 Example. Let $X = \mathrm{Spec}(R)$ and let $Y \subseteq |\,\mathrm{Spec}(R)|$ be a closed subset. Then there exists an ideal $I \subseteq R$ such that $V(I) = Y$, and one has natural isomorphisms

$$\widehat{X}_Y \cong \mathrm{Spf}(R, I) \cong \mathrm{Spf}(R_I^{\wedge}).$$

II.7.0.5 Definition. A formal scheme is *algebraizable* if it is of the form $\widehat{X}_Y$ for some noetherian scheme X and a closed subset $Y \subseteq |X|$.

II.7.0.6 Remark. In the rest of this section, we will restrict our attention on algebraizable formal schemes. Even in the noetherian case, there exist non-algebraizable formal schemes, although they are difficult to construct; see [HM68, Section 5].

II.7.0.7 Theorem. *Let X be a noetherian scheme, and let $Z \subseteq X$ be a closed subset. Let $\widehat{X}_Z$ denote the formal completion of X along Z. Then there is an isomorphism of locally ringed spaces (and hence formal schemes)*

$$\Phi\colon\ \mathrm{Spf}\big(\mathrm{D}_{\mathrm{qc}}(X)^c, Z\big) \xrightarrow{\sim} \widehat{X}_Z.$$

Proof. Let $\{U_\alpha\}$ be a finite affine open cover of X, where $U_\alpha = \mathrm{Spec}(R_\alpha)$. Since U_α is affine, it is a quasi-compact open subset of X. Let $Z_\alpha \coloneqq Z \cap U_\alpha$ be the restriction of the closed subset to the affine patch; note that $Z_\alpha = V(I_\alpha)$ for some ideal $I_\alpha \subseteq R_\alpha$.

By the definition of the formal completion of a scheme, $\widehat{X}_Z$ is obtained by gluing the affine formal schemes

$$(\widehat{X}_Z)|_{U_\alpha} \cong \mathrm{Spf}(R_\alpha, I_\alpha).$$

On the tensor-triangular side, we identify $\mathrm{Spc}(\mathrm{D}_{\mathrm{qc}}(X)^c) \cong X$ as topological spaces (Example II.3.0.20). Since U_α is a quasi-compact open set, we may apply the locality theorem (Theorem II.5.0.16):

$$\mathrm{Spf}\big(\mathrm{D}_{\mathrm{qc}}(X)^c, Z\big)\big|_{Z_\alpha} \cong \mathrm{Spf}\big(\mathrm{D}_{\mathrm{qc}}(X)^c(U_\alpha), Z_\alpha\big).$$

where $Z_\alpha \coloneqq Z \cap U_\alpha$. Recall from Remark II.3.0.10 that $\mathrm{D}_{\mathrm{qc}}(X)^c(U_\alpha)$ denotes the Karoubi quotient. By the Thomason–Trobaugh localization theorem [TT90a, Theorem 5.2.2] (see also [Bal02, Theorem 2.13]), there is an equivalence

$$\mathrm{D}_{\mathrm{qc}}(X)^c(U_\alpha) \simeq \mathrm{D}_{\mathrm{qc}}(U_\alpha)^c \simeq \mathrm{D}(R_\alpha)^c.$$

Thus, we have an isomorphism of formal schemes

$$\mathrm{Spf}\big(\mathrm{D}_{\mathrm{qc}}(X)^c, Z\big)\big|_{Z_\alpha} \cong \mathrm{Spf}\big(\mathrm{D}(R_\alpha)^c, V(I_\alpha)\big).$$

Now we apply the affine formal Hopkins–Neeman theorem (Theorem II.6.0.3). This provides an isomorphism of ringed spaces

$$\rho_\alpha\colon\ \mathrm{Spf}\big(\mathrm{D}(R_\alpha)^c, V(I_\alpha)\big) \xrightarrow{\sim} \mathrm{Spf}(R_\alpha, I_\alpha).$$

Combining these identifications, we obtain local isomorphisms

$$\phi_\alpha\colon\ \mathrm{Spf}\big(\mathrm{D}_{\mathrm{qc}}(X)^c, Z\big)\big|_{Z_\alpha} \xrightarrow{\sim} (\widehat{X}_Z)|_{U_\alpha}.$$

To verify that these glue, consider an intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Since X is qcqs, this intersection is quasi-compact. The comparison map ρ in Theorem II.6.0.3 is natural with respect to localization, and the locality theorem is functorial with respect to restriction. Consequently, the isomorphisms ϕ_α and ϕ_β satisfy the cocycle condition on the overlaps $U_{\alpha\beta}$. Therefore, they glue to a global isomorphism of ringed spaces. The argument is then completed the same way as in Theorem II.6.0.3. □

II.7.0.8 Remark. One can ask for extensions beyond schemes (e.g., to certain algebraic stacks or other locally affine tensor-triangular situations). Conceptually, the argument above only uses (i) an affine identification (a formal Hopkins–Neeman statement), (ii) a locality statement for Spf under restriction to quasi-compact opens, via Theorem II.5.0.16 and (iii) a Thomason–Trobaugh type localization theorem identifying the relevant Verdier quotients with the corresponding "affine" categories on opens. Any setting in which analogues of (i)–(iii) hold admits the same gluing argument.

II.7.0.9 Definition. Let $\widehat{X}_Z$ denote the formal completion of X along Z as above. We set

$$\mathrm{D}^{\wedge}_{\mathrm{qc}}(\widehat{X}_Z) \coloneqq \widehat{\mathrm{D}_{\mathrm{qc}}(X)}_Z,$$

the completion of $\mathrm{D}_{\mathrm{qc}}(X)$ at the Thomason subset $Z \subseteq \mathrm{Spc}(\mathrm{D}_{\mathrm{qc}}(X)^c) \cong X$ in the sense of Definition II.4.0.3.

II.7.0.10 Corollary. *Let X be a noetherian scheme and let $Z \subseteq X$ be a closed subset. Let $\widehat{X}_Z$ denote the formal completion of X along Z. Then there is an isomorphism of ringed spaces (and hence formal schemes)*

$$\Phi\colon\ \mathrm{Spf}\big(\mathrm{D}^{\wedge}_{\mathrm{qc}}(\widehat{X}_Z)^{\mathrm{d}}\big) \xrightarrow{\sim} \widehat{X}_Z.$$

II.7.0.11 Remark. Recall from Remark II.4.0.4 that completion at a Thomason subset $Z \subseteq \mathrm{Spc}(\mathrm{D}_{\mathrm{qc}}(X)^c) \cong X$ is given by the endofunctor

$$\widehat{(-)}_Z \simeq \mathsf{hom}(e_Z, -),$$

where e_Z is the Balmer–Favi idempotent associated to Z.

When X is separated, one can identify this idempotent with a familiar geometric object: there is an equivalence

$$e_Z \simeq \mathbb{R}\Gamma_Z(\mathcal{O}_X)$$

in $\mathrm{D}_{\mathrm{qc}}(X)$, where $\mathbb{R}\Gamma_Z$ denotes the right derived functor of Γ_Z, the subsheaf with supports in Z, see [Ste18, Lemma 5.4]. In particular, in this case the abstract completion functor $\mathsf{hom}(e_Z, -)$ agrees with the derived completion (local homology) functor of Alonso Tarrío–Jeremías López–Lipman: by [AJL97, Theorem (0.3)], the right adjoint to $\mathbb{R}\Gamma_Z$ is naturally isomorphic to $\mathbb{L}\Lambda_Z$, and under the identification $e_Z \simeq \mathbb{R}\Gamma_Z(\mathcal{O}_X)$ this yields a natural equivalence of endofunctors on $\mathrm{D}_{\mathrm{qc}}(X)$

$$\widehat{(-)}_Z \simeq \mathsf{hom}(\mathbb{R}\Gamma_Z(\mathcal{O}_X), -) \simeq \mathbb{L}\Lambda_Z(-).$$

II.7.0.12 Remark. We briefly indicate how Definition II.7.0.9 compares with the triangulated category denoted $\mathrm{D}^{\wedge}_{\mathrm{qc}}(\widehat{X}_Z)$ in [AJL99, §6.3], at least when X is separated. This comparison is not used elsewhere in the paper.

Write $\widetilde{\mathrm{D}}^{\wedge}_{\mathrm{qc}}(\widehat{X}_Z)$ for the category constructed in [AJL99, Remark 6.3.1]. Alonso–Jeremías–Lipman show that on the formal scheme $\widehat{X}_Z$ the "complete" and "torsion" variants agree, and identify the torsion category on $\widehat{X}_Z$ with the Z-supported subcategory on X:

$$\widetilde{\mathrm{D}}^{\wedge}_{\mathrm{qc}}(\widehat{X}_Z) \simeq \mathrm{D}_{\mathrm{qct}}(\widehat{X}_Z) \simeq \mathrm{D}_{\mathrm{qc}\,Z}(X),$$

where $\mathrm{D}_{\mathrm{qc}\,Z}(X)$ denotes the essential image of $\mathbb{R}\Gamma_Z$ on $\mathrm{D}_{\mathrm{qc}}(X)$. (For precise statements and hypotheses, see [AJL99, §6.3] and [AJL99, Proposition 5.2.4].)

On the other hand, Remark II.7.0.11 identifies our completion functor $\widehat{(-)}_Z$ with $\mathbb{L}\Lambda_Z$ (when X is separated). A form of Greenlees–May duality identifies the torsion and complete pieces as equivalent subcategories of $\mathrm{D}_{\mathrm{qc}}(X)$, so the essential image of $\mathbb{L}\Lambda_Z$ agrees with the completion subcategory $\widehat{\mathrm{D}_{\mathrm{qc}}(X)}_Z$ of Definition II.7.0.9. Under these identifications, one obtains an equivalence

$$\widetilde{\mathrm{D}}^{\wedge}_{\mathrm{qc}}(\widehat{X}_Z) \simeq \widehat{\mathrm{D}_{\mathrm{qc}}(X)}_Z.$$

In fact, the formulas in [AJL99, Proposition 5.24] implies that this equivalence is given by $\mathbb{L}\Lambda_Z \circ k_*$ where $k\colon \widehat{X}_Z \to X$ is the completion map.

Chapter II.8

A comparison map

One of the most important tools in computing the Balmer spectrum of a tensor-triangulated category is Balmer's comparison map (Theorem II.3.0.22). A general strategy for computing $\mathrm{Spc}(\mathcal{K})$ is to first determine $\mathrm{Spec}^h(\pi_* \mathrm{Hom}_{\mathcal{K}}(\mathbb{1}, \mathbb{1}))$ and then compute the fibers of the comparison map; see [BS17] and [Aro+25] for examples. It is therefore natural to seek an analogue of the comparison map for the formal spectrum introduced in this paper. Throughout this section we assume $R_* := \pi_* \mathrm{Hom}_{\mathcal{K}}(\mathbb{1}, \mathbb{1})$ is noetherian, which implies that ρ is surjective [Bal10, Theorem 7.3]. In this section we will construct a comparison map for 'algebraic' completions, i.e., those coming from an ideal in R_*.

II.8.0.1 Definition. Let $\mathcal{K}$ be a 2-Ring and let $I \subseteq R_*$ be a homogeneous ideal. Write $Y(I) := \rho^{-1}(V(I))$ for the preimage of the closed subset $V(I) \subseteq \mathrm{Spec}^h(R_*)$ under the comparison map of Theorem II.3.0.22. Define

$$\widehat{\mathcal{C}}_I := \widehat{\mathcal{C}}_{Y(I)}$$

to be the completion of $\mathcal{C}$ at the closed subset $Y(I) \subseteq \mathrm{Spc}(\mathcal{K})$ in the sense of Definition II.4.0.3.

II.8.0.2 Example. If $\mathcal{K} = \mathrm{Perf}(R)$ for a commutative noetherian ring

R, then ρ is a homeomorphism, and we are exactly in the situation of Example II.4.0.6.

II.8.0.3 Remark. The spectrum $R \coloneqq \mathrm{Hom}_{\mathcal{C}}(\mathbb{1},\mathbb{1})$ is a commutative algebra in spectra and, for any $X \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X)$ is an R-module. By Morita theory there is a symmetric monoidal adjunction

$$F\colon \mathrm{Mod}_R \rightleftarrows \mathcal{C} : G \tag{II.8.0.4}$$

with $F(M) = M \otimes_R \mathbb{1}$ and $G(X) = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X)$. Since $\mathcal{C} = \mathrm{Ind}(\mathcal{K})$ is rigidly-compactly generated and $\mathbb{1}$ is compact, F is fully faithful and G exhibits Mod_R as a colocalization of $\mathcal{C}$.

Let $\rho_{\mathcal{K}}\colon \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec}^h(R_*)$ and $\rho_R\colon \mathrm{Spc}(\mathrm{Mod}_R^c) \to \mathrm{Spec}^h(R_*)$ be the respective comparison maps. Functoriality of the comparison map [Bal10, Theorem 5.3] yields a commutative diagram

$$\begin{tikzcd} \mathrm{Spc}(\mathcal{K}) \arrow[r, "\mathrm{Spc}(F)"] \arrow[d, two heads, "\rho_{\mathcal{K}}"'] & \mathrm{Spc}(\mathrm{Mod}_R^c) \arrow[d, two heads, "\rho_R"] \\ \mathrm{Spec}^h(R_*) \arrow[r, equal] & \mathrm{Spec}^h(R_*) \end{tikzcd}$$

and hence, for any homogeneous ideal $I \subseteq R_*$, the associated Thomason subsets

$$Y(I) \coloneqq \rho_{\mathcal{K}}^{-1}\big(V(I)\big) \subseteq \mathrm{Spc}(\mathcal{K}), \qquad Y_R(I) \coloneqq \rho_R^{-1}\big(V(I)\big) \subseteq \mathrm{Spc}(\mathrm{Mod}_R^c)$$

satisfy

$$Y(I) = \mathrm{Spc}(F)^{-1}\big(Y_R(I)\big).$$

Let $e_{Y(I)} \in \mathcal{C}$ and $e_{Y_R(I)} \in \mathrm{Mod}_R$ be the associated tensor idempotents. By functoriality [BS17, Proposition 5.11] we have $F(e_{Y_R(I)}) \cong e_{Y(I)}$, and since F is fully faithful, $G(e_{Y(I)}) \cong e_{Y_R(I)}$ in Mod_R.

II.8.0.5 Proposition. *For every $X \in \mathcal{C}$ there is a natural isomorphism of R-modules*

$$\mathrm{Hom}_{\mathcal{C}}(e_{Y(I)}, X) \cong \mathrm{Hom}_{\mathrm{Mod}_R}\big(e_{Y_R(I)}, \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X)\big).$$

Proof. Using $F(e_{Y_R(I)}) \cong e_{Y(I)}$ and the adjunction (II.8.0.4),

$$\mathrm{Hom}_{\mathcal{C}}(e_{Y(I)}, X) \cong \mathrm{Hom}_{\mathcal{C}}\big(F(e_{Y_R(I)}), X\big) \cong \mathrm{Hom}_{\mathrm{Mod}_R}\big(e_{Y_R(I)}, \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X)\big)$$

as required. □

II.8.0.6 Proposition. *Let $\mathcal{K} \in$ 2-Ring with R_* noetherian, and let $I \subseteq R_*$ be a homogeneous ideal. Set $Y(I) := \rho^{-1}(V(I)) \subseteq \mathrm{Spc}(\mathcal{K})$, where $\rho\colon \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec}^h(R_*)$ is Balmer's comparison map of Theorem II.3.0.22. Then there is a natural morphism of locally ringed spaces*

$$\overline{\rho}\colon \mathrm{Spf}(\mathcal{K}, Y(I)) \longrightarrow \mathrm{Spf}^h(\pi_* R, I),$$

whose underlying map of spaces is the restriction $\rho|_{Y(I)}\colon Y(I) \to V(I)$. In particular, if ρ is an isomorphism, then so is $\overline{\rho}$.

Proof. On spaces, $\rho(Y(I)) = \rho(\rho^{-1}(V(I))) = V(I)$. For the sheaves, it suffices to construct compatible maps on a basis of opens of $\mathrm{Spf}^h(R_*, I)$, namely $\overline{D}(s) = D(s) \cap V(I)$ with $s \in R_*$. By [Bal10, Theorem 5.3(b)], $\rho^{-1}(D(s)) = U(\mathrm{cone}(s))$, hence

$$\overline{\rho}^{-1}\big(\overline{D}(s)\big) = Y(I) \cap \rho^{-1}(D(s)) = \overline{U}(\mathrm{cone}(s)).$$

By definition, $\mathcal{O}_{\mathrm{Spf}^h(R_*,I)}(\overline{D}(s)) = R_*[1/s]^{\wedge}_I$. On the other hand, by Remark II.5.0.3 and Proposition II.8.0.5,

$$\mathcal{O}_{\mathrm{Spf}(\mathcal{K},Y(I))}\big(\overline{U}(\mathrm{cone}(s))\big) \cong \pi_* \mathrm{Hom}_{\mathrm{Mod}_R}\big(e_{Y_R(I)}, \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}[1/s])\big).$$

The right-hand side is the derived I-adic completion of the R-module $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}[1/s])$, computed by the Greenlees–May spectral sequence [GM95, Eq. (3.3), Theorem 4.2] with

$$E_2^{p,q} = H^I_{-p,-q}\big(\pi_* \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}[1/s])\big)$$

Note that $H^I_{-p,*}(-)$ denote the local homology groups of [GM92]. Since $\pi_* \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}[1/s]) \cong R_*[1/s]$ is R_*-flat, [GM92, Theorem 2.5 and 4.1] imply that these terms vanish for $p \neq 0$, so that the spectral sequence collapses and

$$\mathcal{O}_{\mathrm{Spf}(\mathcal{K},Y(I))}\big(\overline{U}(\mathrm{cone}(s))\big) \cong R_*[1/s]_I^\wedge = \mathcal{O}_{\mathrm{Spf}^h(R_*,I)}(\overline{D}(s)).$$

These identifications are natural in s and compatible with restriction, hence glue to $\overline{\rho}^\# \colon \mathcal{O}_{\mathrm{Spf}^h(R_*,I)} \to \overline{\rho}_* \mathcal{O}_{\mathrm{Spf}(\mathcal{K},Y(I))}$. If ρ is an isomorphism, the same local identifications show $\overline{\rho}$ is an isomorphism of ringed spaces. ☐

II.8.0.7 Corollary. *Under the hypotheses above, there is a natural morphism of ringed spaces*

$$\widehat{\rho}\colon\ \mathrm{Spf}\big(\widehat{\mathcal{C}}_Y^{\mathrm{d}}\big) \longrightarrow \mathrm{Spf}^h\big((R_*)_I^\wedge\big).$$

Proof. Combine Theorem II.5.0.14 with Remark II.2.0.5 and Proposition II.8.0.6. ☐

II.8.0.8 Example. Let A be an even periodic commutative ring spectrum with $\pi_0(A)$ a complete regular local ring with maximal ideal $\mathfrak{m}$. By [DS16, Theorem 1.1] the comparison map

$$\rho\colon\ \mathrm{Spc}(\mathrm{Perf}(A)) \longrightarrow \mathrm{Spec}^h(\pi_* A) \cong \mathrm{Spec}(\pi_0 A)$$

is a homeomorphism. In fact, by [Bal10, Proposition 6.11], this is even an isomorphism of ringed spaces. Let $\kappa(A) \coloneqq A/\mathfrak{m}$ (so that $\pi_0\kappa(A) \cong \pi_0(A)/\mathfrak{m}$). This is an A-module with $\mathrm{supp}(\kappa(A)) = V(\mathfrak{m})$. Applying Proposition II.8.0.6 and Corollary II.8.0.7 with $I = \mathfrak{m}$ we obtain a homeomorphism

$$\mathrm{Spf}\big((L_{\kappa(A)}\,\mathrm{Mod}_A)^{\mathrm{d}}\big) \xrightarrow{\cong} \mathrm{Spf}(\pi_0 A).$$

Here we use Remark II.4.0.5 to identify the completion functor as a Bousfield localization in this case.

Chapter II.9

Chromatic homotopy

We now turn to examples coming from chromatic homotopy theory. As usual, we fix a prime number p throughout.

II.9.0.1 Remark. Recall that the Brown–Peterson spectrum BP has coefficient ring

$$BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots].$$

Johnson–Wilson theory $E(n)$ is obtained from BP by first quotienting by the ideal $(v_{n+1}, v_{n+2}, \ldots)$ and then inverting v_n (using, for example, highly structured ring spectra [Elm+97a]). Its coefficients are

$$E(n)_* \cong v_n^{-1} BP_*/(v_{n+1}, v_{n+2}, \ldots).$$

More generally, for an invariant regular sequence $J_k = (p^{i_0}, v_1^{i_1}, \ldots, v_{k-1}^{i_{k-1}})$, we can form a quotient BP/J_k with $\pi_*(BP/J_k) \cong BP_*/J_k$, and for $0 \le n \le k$ define a spectrum $E(n, J_k)$ with

$$E(n, J_k)_* \;\cong\; v_n^{-1}\big((BP/J_k)_*/(v_{n+1}, v_{n+2}, \ldots)\big).$$

For $k = 0$ this recovers $E(n)$, while for $J_k = (p, v_1, \ldots, v_{n-1})$ we have $E(n, J_k) \simeq K(n)$, Morava K-theory. Moreover, by [Hea23, Proposition 2.10], there is an equivalence of Bousfield classes

$$\langle E(n, J_k)\rangle = \langle K(k) \vee \cdots \vee K(n)\rangle,$$

hence

$$\mathcal{S}p_{E(n,J_k)} \simeq \mathcal{S}p_{K(k)\vee\cdots\vee K(n)}.$$

II.9.0.2 Notation. Write $\mathcal{S}p_n \coloneqq \mathcal{S}p_{E(n)}$ for the $E(n)$-local stable homotopy category, $\mathcal{S}p_n^c$ for its subcategory of compact objects, and L_n for Bousfield localization at $E(n)$. For $h \geq 0$ let $K(h)$ denote Morava K-theory and fix a finite type $(h+1)$-spectrum $F(h+1)$.

II.9.0.3 Definition. For $0 \leq h \leq n$, define the $\otimes$-ideal

$$\mathcal{P}_h \coloneqq \left\{ x \in \mathcal{S}p_n^c \,\middle|\, K(h)_*(x) = 0 \right\}.$$

II.9.0.4 Remark. Since $K(h)_*(x \otimes y) \cong K(h)_*(x) \otimes_{K(h)_*} K(h)_*(y)$, each $\mathcal{P}_h$ is prime. Moreover,

$$\mathcal{P}_h = \operatorname{thickid}\langle L_n F(h+1)\rangle.$$

II.9.0.5 Theorem (Hovey–Strickland)**.** *The spectrum of the $E(n)$-local category is the finite chain*

$$\operatorname{Spc}(\mathcal{S}p_n^c) = \{ \underbrace{\mathcal{P}_0}_{\textit{generic}} \subsetneq \mathcal{P}_1 \subsetneq \cdots \subsetneq \underbrace{\mathcal{P}_n}_{\textit{closed}} \}.$$

The topology is defined by the closure operator $\overline{\{\mathcal{P}_h\}} = \left\{ \mathcal{P}_k \,\middle|\, h \leq k \leq n \right\}$. For $0 \leq k \leq n$, consider the basic open set defined by the type $(k+1)$-spectrum $F(k+1)$:

$$U_k \coloneqq U\big(L_n F(k+1)\big) = \left\{ \mathcal{P}_i \,\middle|\, L_n F(k+1) \in \mathcal{P}_i \right\} = \left\{ \mathcal{P}_i \,\middle|\, 0 \leq i \leq k \right\}.$$

The value of the structure sheaf on this open set is

$$\mathcal{O}_{\operatorname{Spc}(\mathcal{S}p_n^c)}(U_k) \;\cong\; \pi_* L_k S^0.$$

Proof. The topological description is due to [HS99, Theorem 6.9]; see also [BHN22, Proposition 3.5] for a translation into this language.

For the sheaf, note that by [HS99, Proposition 6.10] the finite localization of $\mathcal{S}p_n$ away from $L_nF(k+1)$ identifies the localized category $\mathcal{S}p_n[L_nF(k+1)^{-1}]$ with $\mathcal{S}p_k$. Under this identification the localized tensor unit corresponds to L_kS^0. Therefore,

$$\mathcal{O}_{\mathrm{Spc}(\mathcal{S}p_n^c)}(U_k) \cong \pi_* \operatorname{End}_{\mathcal{S}p_k}(L_kS^0) \cong \pi_* L_kS^0. \qquad \square$$

II.9.0.6 Definition. For a fixed height $h < n$, let Y_{h+1} be the closed subset corresponding to spectra of type $\geq h+1$:

$$Y_{h+1} \coloneqq \operatorname{supp}(L_nF(h+1)) = \{\mathcal{P}_{h+1}, \ldots, \mathcal{P}_n\}.$$

II.9.0.7 Remark. By Remark II.4.0.5, the completion $(\widehat{\mathcal{S}p_n})_{Y_{h+1}}$ is equivalent to Bousfield localization of $\mathcal{S}p_n$ at $L_nF(h+1)$. By [BHV18, Lemma 6.16], this is equivalent to localization of $\mathcal{S}p$ at $F(h+1) \otimes E(n)$, which by [Hea23, Proposition 2.10] is equivalent to $\mathcal{S}p_{K(h+1)\vee\cdots\vee K(n)}$ (the special case where $h = n-1$ is considered in [BS25, Proposition 8.11]). Moreover, [Hea23, Corollary 2.29] gives

$$L_nF(h+1) \simeq L_{K(h+1)\vee\cdots\vee K(n)}F(h+1).$$

By Theorem II.5.0.14, we therefore obtain an isomorphism of ringed spaces

$$\operatorname{Spf}(\mathcal{S}p_n^{\mathrm{d}}, Y_{h+1}) \cong \operatorname{Spf}\big(\mathcal{S}p^{\mathrm{d}}_{K(h+1)\vee\cdots\vee K(n)}\big).$$

II.9.0.8 Theorem. *For $0 \leq h \leq n-1$, the underlying topological space of the formal spectrum $\operatorname{Spf}\big(\mathcal{S}p^{\mathrm{d}}_{K(h+1)\vee\cdots\vee K(n)}\big)$ is*

$$Y_{h+1} = \{\mathcal{P}_{h+1} \subsetneq \mathcal{P}_{h+2} \subsetneq \cdots \subsetneq \mathcal{P}_n\} \subseteq \operatorname{Spc}(\mathcal{S}p_n^c).$$

For $k \geq h+1$, on the basic open set

$$\overline{U}_k \coloneqq U(L_nF(k+1)) \cap Y_{h+1} = \{\, \mathcal{P}_i \mid h+1 \leq i \leq k \,\},$$

the structure sheaf takes the value[1]

$$\mathcal{O}\big(\overline{U}_k\big) \cong \pi_* L_{K(h+1)\vee\cdots\vee K(k)}S^0.$$

[1] We omit the subscript on the $\mathcal{O}$ for reasons of space.

Proof. The underlying space statement follows from Theorem II.9.0.5 and the definition of Y_{h+1}.

For $k \geq h+1$, the Bousfield class argument in Remark II.9.0.7 shows that

$$\begin{aligned} \mathcal{O}(\overline{U}_k) &\cong \pi_* \operatorname{Hom}_{\mathcal{S}p_n}(L_n S^0, L_{K(h+1)\vee\cdots\vee K(n)}\, L_k S^0) \\ &\cong \pi_* L_{K(h+1)\vee\cdots\vee K(n)} L_k S^0. \end{aligned}$$

It remains to show that

$$L_{K(h+1)\vee\cdots\vee K(n)} L_k S^0 \simeq L_{K(h+1)\vee\cdots\vee K(k)} S^0.$$

By [Hea23, Remark 2.27], there is a pullback square

$$\begin{tikzcd} L_{K(h+1)\vee\cdots\vee K(n)} L_k S^0 \arrow[r] \arrow[d] & L_{K(h+1)\vee\cdots\vee K(k)} L_k S^0 \arrow[d] \\ L_{K(k+1)\vee\cdots\vee K(n)} L_k S^0 \arrow[r] & L_{K(h+1)\vee\cdots\vee K(k)} L_{K(k+1)\vee\cdots\vee K(n)} L_k S^0. \end{tikzcd}$$

Since $\langle E(k)\rangle = \langle K(0)\vee\cdots\vee K(k)\rangle$ and $K(i)\wedge E(k)\simeq 0$ for $i>k$, we have $L_{K(k+1)\vee\cdots\vee K(n)} L_k S^0 \simeq 0$. Hence the bottom-right corner also vanishes, being a further localization of the bottom-left corner. Therefore, the pullback identifies the upper-left corner with the upper-right corner:

$$L_{K(h+1)\vee\cdots\vee K(n)} L_k S^0 \;\simeq\; L_{K(h+1)\vee\cdots\vee K(k)} L_k S^0.$$

Finally, since $\langle K(h+1)\vee\cdots\vee K(k)\rangle \leq \langle E(k)\rangle$, localization at $E(k)$ does not affect $K(h+1)\vee\cdots\vee K(k)$-localization, and we obtain

$$L_{K(h+1)\vee\cdots\vee K(k)} L_k S^0 \simeq L_{K(h+1)\vee\cdots\vee K(k)} S^0.$$

Combining these equivalences yields the claimed identification, and hence

$$\mathcal{O}(\overline{U}_k) \cong \pi_* L_{K(h+1)\vee\cdots\vee K(k)} S^0,$$

as desired. □

II.9.0.9 Remark. Unsurprisingly, the identification of the sheaf of spectra considered in Remark II.5.0.4 takes the form

$$\mathcal{O}\big(\overline{U}_k\big) \cong L_{K(h+1)\vee\cdots\vee K(k)}S^0,$$

Chapter II.10

Further examples

In this short section, we consider two further examples, from equivariant homotopy theory and modular representation theory.

II.10.0.1 Notation. Let G be a finite group and p a prime. We write $\mathcal{S}p_{G,(p)}^{\mathrm{cofree}}$ for the Bousfield localization of $\mathcal{S}p_{G,(p)}$ at G_+, or equivalently, the completion of $\mathcal{S}p_{G,(p)}$ at the Thomason subset

$$Y_G \coloneqq \operatorname{supp}(G_+) \subseteq \operatorname{Spc}(\mathcal{S}p_{G,(p)}^c).$$

As a set, the spectrum $\operatorname{Spc}(\mathcal{S}p_{G,(p)}^c)$ and the subset Y_G are computed in [BS17]. Moreover, [BS17, Corollary 4.13] shows that restriction induces a homeomorphism from $\operatorname{Spc}(\mathcal{S}p_{(p)}^c)$ onto Y_G.

II.10.0.2 Proposition. *Let G be a finite group and p a prime. Restriction along the inclusion $e \leq G$ defines an exact symmetric monoidal functor*

$$\operatorname{res}_e^G \colon \mathcal{S}p_{G,(p)} \longrightarrow \mathcal{S}p_{(p)}.$$

Let $Y_G \coloneqq \operatorname{supp}(G_+) \subseteq \operatorname{Spc}(\mathcal{S}p_{G,(p)}^c)$. Then res_e^G induces a morphism of ringed spaces

$$(\varphi, \varphi^\#) \colon \operatorname{Spc}(\mathcal{S}p_{(p)}^c) \longrightarrow \operatorname{Spf}(\mathcal{S}p_{G,(p)}^{\mathrm{cofree,d}}).$$

Moreover, φ is a homeomorphism of the underlying topological spaces. However, for $G = C_2$ and $p = 2$, this morphism is not an isomorphism of ringed spaces.

Proof. Apply Lemma II.5.0.9 to the restriction functor between compact objects:
$$\mathrm{res}^G_e \colon \mathcal{S}p^c_{G,(p)} \longrightarrow \mathcal{S}p^c_{(p)}.$$
Let $Y \coloneqq \mathrm{Spc}(\mathcal{S}p^c_{(p)})$. Then $\mathrm{Spc}(\mathrm{res}^G_e)$ is a continuous map
$$\mathrm{Spc}(\mathrm{res}^G_e) \colon \mathrm{Spc}(\mathcal{S}p^c_{(p)}) \longrightarrow \mathrm{Spc}(\mathcal{S}p^c_{G,(p)}),$$
and we set $Y' \coloneqq \mathrm{Spc}(\mathrm{res}^G_e)(Y) \subseteq \mathrm{Spc}(\mathcal{S}p^c_{G,(p)})$. By [BS17, Corollary 4.13], this image is precisely
$$Y' = \mathrm{supp}(G_+) = Y_G.$$
Therefore, Lemma II.5.0.9 yields a morphism of ringed spaces
$$(\varphi, \varphi^{\#}) \colon \mathrm{Spf}(\mathcal{S}p^c_{(p)}, Y) \longrightarrow \mathrm{Spf}(\mathcal{S}p^c_{G,(p)}, Y_G).$$
By Example II.5.0.7, the domain is $\mathrm{Spf}(\mathcal{S}p^c_{(p)}, Y) \cong \mathrm{Spc}(\mathcal{S}p^c_{(p)})$, and by Theorem II.5.0.14, the target identifies with $\mathrm{Spf}(\mathcal{S}p^{\mathrm{cofree,d}}_{G,(p)})$, yielding the stated map. The fact that φ is a homeomorphism on underlying spaces is exactly [BS17, Corollary 4.13].

Finally, we show this is not an isomorphism for $G = C_2$ and $p = 2$ by evaluating global sections. On the nonequivariant side (the source), we have
$$\mathcal{O}_{\mathrm{Spc}(\mathcal{S}p^c_{(2)})}\big(\mathrm{Spc}(\mathcal{S}p^c_{(2)})\big) \cong \pi_* \mathrm{End}_{\mathcal{S}p_{(2)}}(\mathbb{S}_{(2)}) \cong \pi_* \mathbb{S}_{(2)}.$$
In degree 0, this is the 2-local integers $\mathbb{Z}_{(2)}$.

On the cofree side (the target), using Remark II.5.0.6 (and the identification of the whole space with Y_G), we have
$$\mathcal{O}_{\mathrm{Spf}(\mathcal{S}p^{\mathrm{cofree}}_{C_2,(2)})}(Y_G) \cong \pi_* \mathrm{Hom}_{\mathcal{S}p_{C_2,(2)}}(e_{Y_G}, \mathbb{S}_{(2)}).$$

The Segal conjecture for C_2 (specifically Lin's theorem [Lin80, Theorem 1.1]) identifies the degree 0 part of this ring with the I-adic completion of the Burnside ring $A(C_2)$, where I is the augmentation ideal. Since the two rings are not isomorphic (for example, the latter contains the 2-adic integers $\mathbb{Z}_2$ as a subring, and so is uncountable), the induced map on degree-0 global sections is not an isomorphism. Therefore, $(\varphi, \varphi^{\#})$ is not an isomorphism of ringed spaces. $\square$

II.10.0.3 Remark. Our final example comes from modular representation theory. Fix a finite group G and a field k of characteristic p dividing $|G|$. Let $\mathcal{T} := K(\mathrm{Inj}\, kG)$ be the homotopy category of complexes of injective kG-modules (equivalently, one may model the corresponding stable homotopy theory by the ∞-category of modules $\mathcal{T} \simeq \mathrm{Mod}_{\mathcal{S}p_G}(\underline{k})$, where $\underline{k}$ denotes the Borel-equivariant G-spectrum associated to k.). This is a symmetric monoidal, rigidly-compactly generated category, and its compact/dualizable objects identify with the bounded derived category:

$$\mathcal{T}^{\mathrm{d}} \simeq \mathrm{D}^b(\mathrm{Mod}\, kG).$$

By [BCR97] and [Bal10, Proposition 8.5], the comparison map

$$\rho\colon \operatorname{Spc}\big(\mathrm{D}^b(\mathrm{Mod}\, kG)\big) \longrightarrow \operatorname{Spec}^h\big(H^\bullet(G,k)\big)$$

is a homeomorphism. The homogeneous spectrum $\operatorname{Spec}^h(H^\bullet(G,k))$ has a unique closed point, corresponding to the homogeneous maximal ideal $\mathfrak{m} := H^{>0}(G,k)$. Let $Y_\mathfrak{m} \subseteq \operatorname{Spc}(\mathrm{D}^b(\mathrm{Mod}\, kG))$ denote the corresponding singleton.

The tt-completion of $\mathcal{T}$ at $Y_\mathfrak{m}$ identifies with the unbounded derived category $\mathrm{D}(\mathrm{Mod}\, kG)$ equipped with the diagonal tensor product; see [BS25, Example 3.29]. In this category, the dualizable objects are again $\mathrm{D}^b(\mathrm{Mod}\, kG)$, although the compact objects need not coincide with the dualizable ones. In particular, the completed category is typically not rigidly-compactly generated. Using Proposition II.8.0.6 we deduce the following:

II.10.0.4 Theorem. *With notation as above, the formal spectrum* $\mathrm{Spf}\big(\mathrm{D}^b(\mathrm{Mod}\, kG)\big)$ *is a one-point ringed space, and its ring of global sections is* $H^\bullet(G,k)^\wedge_{\mathfrak{m}}$.

Part III

On filtered stacks and the geometrization of synthetic spectra

On filtered stacks and the geometrization of synthetic spectra

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Abstract. We introduce the notion of "filtered stacks", in order to generalize the theory of synthetic spectra to spectral algebraic geometry. Specifically, we show that any spectral stack admits an associated filtered stack X^τ and we prove that quasi-coherent sheaves on X^τ are a 1-parameter deformation of quasi-coherent sheaves on X. Furthermore, we show that the special fiber of this deformation is quasi-coherent sheaves on $X^{d\heartsuit}$, the underlying Dirac stack of X. In particular, this allows us to extend the "cofiber of τ" method to the study of descent spectral sequences.

The paper has three main parts. We first develop the affine theory, where we study commutative algebras in the category of filtered spectra and prove a version of faithfully flat descent in this setting. We then develop the theory of filtered stacks, focusing on geometric filtered stacks and their categories of quasi-coherent sheaves. Here we show that any geometric spectral stack is sent to a geometric filtered stack, and that coherent cohomology of the resulting filtered stack refines to a filtered spectrum representing the descent spectral sequence.

Finally, in the last part, we focus on two applications. First, we construct a comparison functor from E-based synthetic spectra to quasi-coherent sheaves on a specific filtered stack $\mathcal{M}_E^\tau$. This comparison is not an equivalence in general, but we show that it is, in a precise sense, controlled by the E-based Adams spectral sequence. Second, we construct a colimit preserving functor from spectral stacks to filtered stacks, which on affines, encodes the *even filtration* of Hahn–Raksit–Wilson. Based on this we define an even filtration associated to any spectral stack.

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Chapter III.1

Introduction

Let E be a commutative ring spectrum of Adams-type, so that the pair (E_*, E_*E) forms a Hopf algebroid. If we let $\mathcal{M}_{E_*}$ denote the stack presented by the Hopf algebroid (E_*, E_*E), then for any spectrum X the $\mathbb{E}_2$-page of the E-based Adams spectral sequence of X takes the form:

$$H^*(\mathcal{M}_{E_*}, \mathcal{F}_{E_*X}) \implies \pi_*(X^\wedge_E).$$

Here $\mathcal{F}_{E_*X}$ is the quasi-coherent sheaf identified with the (E_*, E_*E)-comodule E_*X, via the equivalence

$$\mathrm{QCoh}(\mathcal{M}_{E_*})^\heartsuit \simeq \mathrm{Comod}_{E_*E}$$

of abelian categories, see [HP24, Section 5.1] for a review. Since E is Adams-type, we have that $E_*E = \pi_*(E \otimes E)$ is flat as a Dirac ring over E_*, and we can consider the E-based Adams spectral sequence of X as the descent spectral sequence of a quasi-coherent sheaf $\mathcal{F}_X$ on the "spectral stack" presented by the cosimplicial commutative ring spectrum $E^\bullet$ coming from the Čech nerve of $\mathbb{S} \to E$. Here spectral stack means a stack with respect to the π_*-flat topology on commutative algebra objects in spectra. This is different from the spectral algebraic geometry introduced in [Lurb], since not every π_*-flat map of commutative ring spectra is flat in the sense of Lurie.

A powerful tool for studying the Adams spectral sequence and stable homotopy groups by extension is the so-called "cofiber of τ"-method and the ∞-category of E-based synthetic spectra, introduced by Pstrągowski in [Pst22]. In this paper we introduce a candidate theory for "synthetic quasi-coherent sheaves" on a spectral stack. We do this by introducing a theory of filtered stacks, whose test objects are given by commutative algebra objects in the ∞-category of filtered spectra equipped with the Day convolution symmetric monoidal structure. This theory is set up to be compatible with descent. Specifically, we introduce a flat topology on filtered ring spectra, and prove faithfully flat descent for this topology. One major goal of this paper is to show that the ∞-category of synthetic quasi-coherent sheaves on a spectral stack X deforms the descent spectral sequence of X. This is analogous to how the ∞-category of E-based synthetic spectra is a deformation of the E-based spectral sequence.

More precisely, to a spectral stack X we associate a *filtered stack* X^τ, obtained from the Postnikov filtration on affines via left Kan extension. The associated ∞-category of quasi-coherent sheaves $\mathrm{QCoh}(X^\tau)$ carries a canonical endomorphism

$$\tau\colon \Sigma^{0,-1}\mathcal{O}_{X^\tau} \to \mathcal{O}_{X^\tau}.$$

We similarly construct an associated Dirac stack denoted $X^{d\heartsuit} := ((X^\tau)^\heartsuit)$. The terms in the following theorem are constructed below, but the statement is one of the main results of the paper.

Theorem A (Proposition III.3.3.5, Proposition III.3.3.8, and Theorem III.3.3.16)**.** *If X is any spectral stack, then inverting the endomorphism τ and base-change along $\mathcal{O}_{X^\tau} \to \mathcal{O}_{X^\tau}/\tau$ induce equivalences*

$$\mathrm{QCoh}(X^\tau)[\tau^{-1}] \xrightarrow{\sim} \mathrm{QCoh}(X) \ \textit{and}\ \mathrm{QCoh}(X^\tau)[\tau = 0] \xrightarrow{\sim} \mathrm{QCoh}(X^{d\heartsuit}).$$

Furthermore, if X is a geometric spectral stack, then the Whitehead tower $\tau_{\geqslant \bullet}\colon \mathcal{S}p \to \mathcal{S}p^{\mathrm{fil}}$ induces a lax symmetric monoidal functor

$$\nu_X \colon \operatorname{QCoh}(X) \to \operatorname{QCoh}(X^\tau),$$

such that the diagram

$$\begin{tikzcd}
 & \operatorname{QCoh}(X) \arrow[d, "\nu_X"] \arrow[dl, bend right, "\mathrm{Id}"'] \arrow[dr, bend left, "\pi_*(-)"] & \\
 & \operatorname{QCoh}(X^\tau) \arrow[dl, "\tau^{-1}"] \arrow[dr, "(-)/\tau"'] & \\
\operatorname{QCoh}(X) & & \operatorname{QCoh}(X^{d\heartsuit})
\end{tikzcd}$$

commutes.

Thus $\operatorname{QCoh}(X^\tau)$ is a one-parameter deformation of $\operatorname{QCoh}(X)$, with special fiber $\operatorname{QCoh}(X^{d\heartsuit})$. In the case where X is geometric, the coherent cohomology of X^τ refines to a filtered spectrum representing the descent spectral sequence of X.

We now describe the construction and main results in more detail.

Affine filtered geometry

We begin by defining the flat topology. In more detail, let $\mathcal{S}p^{\mathrm{fil}}$ denote the category of filtered spectra with its Day convolution symmetric monoidal structure. It admits a t-structure, called the Postnikov t-structure, where a filtered spectrum

$$\cdots \to X_2 \to X_1 \to X_0 \to X_{-1} \to X_{-2} \to \dots$$

is connective if the spectrum X_n is n-connective for every n. This t-structure is right and left complete, compatible with filtered colimits and the symmetric monoidal structure on $\mathcal{S}p^{\mathrm{fil}}$. Furthermore, the heart of the Postnikov t-structure is the category of graded abelian groups, with the Koszul tensor product. We denote the t-structure homotopy groups with respect to the Postnikov t-structure by $\pi_k^P\colon \mathcal{S}p^{\mathrm{fil}} \to \mathrm{Ab}^{\mathrm{gr}}$.

Definition III.1.0.1. A map of connective commutative filtered ring spectra $R \to S$ is *flat* if $\pi_0^P R \to \pi_0^P S$ is a flat map of Dirac rings and the canonical map

$$\pi_*^P R \underset{\pi_0^P R}{\otimes} \pi_0^P S \to \pi_*^P S$$

is an equivalence. Furthermore, it is *faithfully* flat if base-change

$$\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}}) \xrightarrow{S \otimes_R -} \mathrm{Mod}_S(\mathcal{S}p^{\mathrm{fil}})$$

is conservative. A finite collection of maps $\{R \to R_i\}_{i \in I}$ is a *flat cover*, if

$$R \to \prod_{i \in I} R_i$$

is faithfully flat.

One class of flat maps of filtered ring spectra comes from π_*-flat maps of commutative ring spectra. More precisely, a map $R \to S$ of commutative ring spectra induces a (faithfully) flat map of Dirac rings on homotopy groups if and only if the map $\tau_{\geqslant \bullet} R \to \tau_{\geqslant \bullet} S$ of connective filtered commutative ring spectra is (faithfully) flat, see Proposition III.2.2.21. We show, following Lurie's proof in the case of the flat topology, that the collection of flat covers determines a finitary Grothendieck topology on $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})^{\mathrm{op}}_{\geqslant 0}$, see Proposition III.2.2.15. In order to do so, we introduce a Künneth spectral sequence for filtered spectra, which we believe may be of independent interest. We further develop the ideas needed for defining the Künneth spectral sequence for filtered spectra in Appendix A.1, where we construct it and other classical spectral sequences in homotopy theory in great generality.

The first major theorem of this paper is a proof of faithfully flat descent for connective filtered ring spectra.

Theorem B (Theorem III.2.4.3)**.** *If $R \to S$ is a faithfully flat map of connective commutative filtered ring spectra, then the functor*

$$\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}}) \to \lim_{[n]\in\Delta} \mathrm{Mod}_{S^{\otimes_R n+1}}(\mathcal{S}p^{\mathrm{fil}})$$

is a symmetric monoidal equivalence of ∞-categories.

A direct consequence of this result is the following descent theorem for spectra:

Theorem C (Theorem III.2.5.3)**.** *If $R \to S$ is a faithfully π_*-flat map of commutative ring spectra, then the natural functor*

$$\mathrm{Mod}_R(\mathcal{S}p) \to \lim_{[n]\in\Delta} \mathrm{Mod}_{S^{\otimes_R n+1}}(\mathcal{S}p)$$

is an equivalence.

Our faithfully flat descent results form the basis for introducing the notion of a filtered stack.

Definition III.1.0.2. We define the ∞-category of *filtered stacks* to be the colimit

$$\mathrm{FStack} := \mathrm{colim}_{\mathcal{C}\subseteq \mathrm{FAff}}\, \mathrm{Shv}(\mathcal{C}),$$

indexed over all small subsites of the site on $\mathrm{FAff} := (\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0})^{\mathrm{op}}$.

From spectral stacks to filtered stacks.

One can define the ∞-category of spectral stacks similarly, and the next theorem shows that any spectral stack produces a filtered stack. More precisely, we prove the following

Theorem D (Theorem III.3.1.9)**.** *There is a colimit preserving functor*

$$(-)^{\tau}\colon \mathrm{Stk} \to \mathrm{FStack},$$

such that for every commutative ring spectrum R we have $\operatorname{Spec}^{\tau} R \coloneqq (\operatorname{Spec} R)^{\tau} \simeq \operatorname{Spec} \tau_{\geqslant \bullet} R$.

The functor $(-)^{\tau}$ is induced by $\tau_{\geqslant \bullet}\colon \mathrm{CAlg}(\mathcal{S}p) \to \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0}$, via left Kan extension. The functor $\tau_{\geqslant \bullet}(-)$ is a map of sites, but it is not left exact. As a result, several good properties of spectral stacks are not preserved by $(-)^{\tau}$. One property of maps that is preserved is being flat.

Theorem E (Corollary III.3.1.29, Proposition III.3.1.26, and Proposition III.3.1.28)**.** *Let*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ X' \arrow[r] & X \end{tikzcd}$$

be a cartesian square of spectral stacks. If p is flat and X is a geometric spectral stack, that is X admits a flat cover by an affine, then $p^{\tau}\colon Y^{\tau} \to X^{\tau}$ is a flat map of filtered stacks, X^{τ} is a geometric filtered stack and

$$\begin{tikzcd} Y'^{\tau} \arrow[r] \arrow[d] & Y^{\tau} \arrow[d, "p"] \\ X'^{\tau} \arrow[r] & X^{\tau} \end{tikzcd}$$

is a cartesian square of filtered stacks.

Here, similarly to ordinary spectral algebraic geometry, we define a map of filtered stacks to be flat if it is representable (by affines) and for every affine mapping to the target, the base-change is a flat map of filtered ring spectra.

Since the functor

$$\pi_0^P\colon \operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0} \to \operatorname{CAlg}(\mathrm{Ab}^{\mathrm{gr}})$$

defines an exact map of sites $\mathrm{FAff} \to \mathrm{DAff}$[1], we obtain a left exact colimit preserving functor

$$(-)^{\heartsuit}\colon \mathrm{FStack} \to \mathrm{DStk},$$

such that if $R \in \operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0}$, then $(\operatorname{Spec} R)^{\heartsuit} \simeq \operatorname{Spec} \pi_0^P R$. We denote the composite $((-)^{\tau})^{\heartsuit} := (-)^{d\heartsuit}$. It follows that for a commutative ring spectrum S, we obtain an equivalence

$$(\operatorname{Spec} S)^{d\heartsuit} = (\operatorname{Spec} \tau_{\geqslant \bullet} S)^{\heartsuit} = \operatorname{Spec} \pi_* S.$$

Inspired by the etale rigidity results of Lurie [Lur17b, Theorem 7.5.4.2], and Hesselholt–Pstrągowski [HP25, Theorem 4.33] we prove the following version of etale rigidity in the filtered setting. We define a map of connective filtered commutative ring spectra to be

Theorem F (Corollary III.2.2.26 and Theorem III.2.2.30)**.** *If $R \in \operatorname{CAlg}(\mathcal{S}p)$ is a commutative ring spectrum, then the functor*

$$\operatorname{CAlg}(\mathcal{S}p)_{R/} \xrightarrow{\tau_{\geqslant \bullet}} \operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})_{\tau_{\geqslant \bullet} R/}$$

restricts to an equivalence on the full subcategories $\operatorname{CAlg}(\mathcal{S}p)^{\flat}_{R/}$ and $\operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})^{\flat}_{\tau_{\geqslant \bullet} R/}$ spanned by π_*-flat R-algebras and flat filtered $\tau_{\geqslant \bullet} R$-algebras. In particular, if we further restrict to etale algebras we obtain equivalences

$$\operatorname{CAlg}(\mathcal{S}p)^{\mathrm{et}}_{R/} \simeq \operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})^{\mathrm{et}}_{\tau_{\geqslant \bullet} R/} \simeq \mathrm{DRing}^{\mathrm{et}}_{\pi_* R/},$$

by etale rigidity [HP25, Theorem 4.33].

[1] Here DAff is given the flat topology introduced by Hesselholt and Pstrągowski in [HP25].

Based on the above result we use descent to show the following result.

Theorem G. *If X is a geometric spectral stack, then the functors*

$$\mathrm{Stk}^{\mathrm{et}}_{/X} \xrightarrow{(-)^{\tau}} \mathrm{FStack}^{\mathrm{et}}_{/X^{\tau}} \xrightarrow{(-)^{\heartsuit}} \mathrm{DStk}^{\mathrm{et}}_{/X^{d\heartsuit}}$$

are equivalence.

Quasi-coherent sheaves.

In Section III.3.2 we define the category of quasi-coherent sheaves on a filtered stack. This theory is Kan extended from the functor $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathcal{S}p^{\mathrm{fil}}/}$ and as such most of the standard results one would expect for such a construction are true. By construction, if X is a filtered stack, then $\mathrm{QCoh}(X)$ admits a canonical $\mathcal{S}p^{\mathrm{fil}}$-module structure in Pr^{L}. In particular, the unit $\mathcal{O}_X$ admits a twisted endomorphism

$$\tau \colon \Sigma^{0,-1}\mathcal{O}_X \to \mathcal{O}_X,$$

induced by the natural transformation

$$\Sigma^{0,-1}(-) \Rightarrow \mathrm{Id}_{\mathcal{S}p^{\mathrm{fil}}}.$$

One remarkable property of this endomorphism is that the cofiber $\mathcal{O}_X/\tau$ admits a canonical commutative algebra structure.

Similarly, there exists a theory of quasi-coherent sheaves on Dirac stacks, studied in [HP24]. We will abuse notation and write $\mathrm{QCoh}(X)$ for the ∞-category of quasi-coherent sheaves on X regardless of whether X is a spectral, filtered or Dirac stack. This abuse is minimal, since it is easy to type-check the stack in question whenever these types of quasi-coherent sheaves appear in the same context.

The remaining part of this paper will be focused on studying the ∞-category of quasi-coherent sheaves on a filtered stack and its interaction with the ∞-category of quasi-coherent sheaves on the underlying spectral stack and associated Dirac stack. The main result is that if X is a geometric spectral stack, then the ∞-category of quasi-coherent sheaves on X^τ is a one-parameter deformation of $\mathrm{QCoh}(X)$, with special fiber $\mathrm{QCoh}(X^{d\heartsuit})$.

If X is a filtered stack, then the $\mathcal{S}p^{\mathrm{fil}}$-module structure on $\mathrm{QCoh}(X)$ also equips mapping spectra with a canonical filtration. In particular, coherent cohomology of X canonically refines to a filtered spectrum $\Gamma(X,-)$. In the case where X comes from a spectral stack we identify this filtered spectrum with the filtered spectrum representing the descent spectral sequence.

Theorem H (Proposition III.3.4.11)**.** *If X is a geometric spectral stack and $\mathcal{F}$ is a quasi-coherent sheaf on X, then there is an equivalence of filtered spectra:*

$$\Gamma(X^\tau, \nu_X \mathcal{F}) \simeq \mathrm{DF}(X, \mathcal{F}),$$

where $DF(X,\mathcal{F})$ is the filtered spectrum representing the descent spectral sequence of $\mathcal{F}$.

Comparison with synthetic spectra.

The remaining part of the paper focuses on two applications. We first develop a strong relationship between the theory of synthetic spectra and quasi-coherent sheaves on certain filtered stacks. Afterwards, we show that the *even filtration* of a commutative ring spectrum R, defined by Hahn–Raksit–Wilson, naturally arises as the coherent cohomology on a filtered stack $\mathrm{Spec}^{\tau-\mathrm{ev}} R$.

If E is an Adams-type commutative algebra object in spectra, then we let Syn_E denote the ∞-category of E-based synthetic spectra as defined by Pstrągowski [Pst22], which serves as a deformation of the ∞-category of spectra and the E-based Adams spectral sequence. The ∞-category of E-based synthetic spectra is a presentably symmetric monoidal stable ∞-category, and admits a canonical t-structure called the *homological* t-structure. Work of Burklund–Hahn–Senger [BHS26, Example C.16] shows that there exists a colimit preserving symmetric monoidal functor

$$\iota\colon \mathcal{S}p^{\mathrm{fil}} \to \mathrm{Syn}_E$$

whose right adjoint we will denote σ following work of Carrick–Davies–Van-Nigtevecht [CDN25]. The ∞-category of E-based synthetic spectra comes with a lax symmetric monoidal functor

$$\nu\colon \mathcal{S}p \to \mathrm{Syn}_E$$

and if M is a homotopy E-module, then $\sigma\nu M \simeq \tau_{\geqslant\bullet} M$ cf. [CDN25, Lemma 1.19]. In particular, σ refines to a lax symmetric monoidal functor

$$\sigma\colon\ \mathrm{Mod}_{\nu E}(\mathrm{Syn}_E) \to \mathrm{Mod}_{\tau_{\geqslant\bullet} E}(\mathcal{S}p^{\mathrm{fil}}), \tag{III.1.0.3}$$

whose composite with $\nu\colon\ \mathrm{Mod}_E(\mathcal{S}p) \to \mathrm{Mod}_{\nu E}(\mathrm{Syn}_E)$ is naturally isomorphic as a lax monoidal functor to $\tau_{\geqslant\bullet}\colon\ \mathrm{Mod}_E(\mathcal{S}p) \to \mathrm{Mod}_{\tau_{\geqslant\bullet} E}(\mathcal{S}p^{\mathrm{fil}})$.

Theorem I (Theorem III.4.2.7)**.** *The functor in Eq. (III.1.0.3) is a symmetric monoidal equivalence.*

This provides our road map to comparing Syn_E with quasi-coherent sheaves on a filtered stack. Let $\mathcal{M}_E \coloneqq |\operatorname{Spec} E^{\otimes\bullet+1}|$ denote the geometric realization of the Čech nerve of $\operatorname{Spec} E \to \operatorname{Spec} \mathbb{S}$, we prove in Corollary III.4.1.6 that $\mathcal{M}_E$ is a geometric spectral stack whenever $E \otimes E$ is π_*-flat over E; this happens for instance when E is Adams-type. In particular, $\mathcal{M}_E^\tau$ is a geometric filtered stack and we have a symmetric monoidal equivalence

$$\mathrm{QCoh}(\mathcal{M}_E^\tau) \xrightarrow{\sim} \lim_{[n]\in\Delta} \mathrm{Mod}_{\tau_{\geqslant\bullet}(E^{\otimes n+1})}(\mathcal{S}p^{\mathrm{fil}}).$$

The above equivalence and Eq. (III.1.0.3) allow us to prove:

Theorem J (Theorem III.4.2.11, Proposition III.4.2.12)**.** *For every Adams-type commutative ring spectrum E, there exists a colimit-preserving symmetric monoidal functor*

$$\mu\colon \operatorname{Syn}_E \to \operatorname{QCoh}(\mathcal{M}_E^\tau)$$

Moreover, if μ^R is its right adjoint, then the composite $\mu^R\mu$ is naturally equivalent to the functor

$$X \mapsto \lim_{[n]\in\Delta} (\nu E^{\otimes n+1} \otimes X),$$

with the evident counit. In particular, μ is fully faithful on the full subcategory of νE-nilpotent complete objects.

This relationship could be studied further, but for now we settle with the following result suggested to us by William Balderrama:

Theorem K (Theorem III.4.2.15)**.** *If $E \in \operatorname{CAlg}(\mathcal{S}p)$ is Adams-type, then the functor*

$$\mu\colon \operatorname{Syn}_E \to \operatorname{QCoh}(\mathcal{M}_E^\tau)$$

identifies $\operatorname{QCoh}(\mathcal{M}_E^\tau)$ as the left completion of Syn_E, with respect to the homological t-structure on Syn_E and the canonical t-structure on $\operatorname{QCoh}(\mathcal{M}_E^\tau)$.

The even filtration

The even filtration of Hahn–Raksit–Wilson defined in [HRW25] measures the failure of a spectrum to have even homotopy groups. This might seem simple, but in practice it agrees with complicated spectral sequences such as the (evenly graded) Adams–Novikov spectral sequence for the sphere or the motivic filtration on topological Hochschild homology. For a ring spectrum R it is defined as

$$\operatorname{fil}^{\mathrm{ev}}_* R \coloneqq \lim_{\substack{R\to S\\ S \text{ even}}} \tau_{\geqslant 2\bullet} S.$$

Hahn–Raksit–Wilson show that the even filtration satisfies descent with respect to the evenly π_*-flat topology on $\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{op}}$, cf. [HRW25, Cor. 2.2.21]. The descent result of Hahn–Raksit–Wilson and its definition as a limit of filtered ring spectra, makes it reasonable to ask whether there exists a filtered stack $(\operatorname{Spec} R)^{\tau-\mathrm{ev}}$,[1] which has the even filtration of R as its filtered cohomology. We prove the following result:

Theorem L (Theorem III.4.4.7, Proposition III.4.3.13)**.** *There exists a colimit preserving functor*

$$(-)^{\tau-\mathrm{ev}} \colon \mathrm{Stk} \to \mathrm{FStack}$$

such that, for every commutative ring spectrum R, there is an equivalence

$$\Gamma((\operatorname{Spec} R)^{\tau-\mathrm{ev}}) \simeq \mathrm{fil}^{\mathrm{ev}}_* R.$$

Moreover, if R is even, then $(\operatorname{Spec} R)^{\tau-\mathrm{ev}} = \operatorname{Spec} \tau_{\geqslant 2\bullet} R$.

The descent results of Hahn–Raksit–Wilson not only play a major role in proving Theorem L, they also provide strong descent results for $(-)^{\tau-\mathrm{ev}}$ and we prove strong interactions between the geometry of spectral stacks and its associated evenly filtered stack. For instance we prove:

Theorem M (Corollary III.4.3.16, Proposition III.4.4.10)**.** *If X is a geometric spectral stack, which admits an evenly flat cover $\operatorname{Spec} R \to X$, where R is even, then $X^{\tau-\mathrm{ev}}$ is a geometric filtered stack and*

$$(\operatorname{Spec} R)^{\tau-\mathrm{ev}} \to X^{\tau-\mathrm{ev}}$$

is a faithfully flat cover.

[1] While this notation suggests that $\operatorname{Spec}^{\tau-\mathrm{ev}} R$ should perhaps be affine, for the purpose of this paper we use a variant which is not affine.

III.1.1 Conventions

This paper is written in the language of ∞-categories, but in order to minimize notational clutter we will simply write category instead for the rest of this paper. Furthermore, we adopt the following conventions:

1. Instead of (commutative) algebra object in spectra we will say (commutative) ring spectrum. Similarly, we will say filtered (commutative) ring spectrum for a (commutative) algebra object in $\mathcal{S}p^{\mathrm{fil}}$, and for a filtered commutative ring spectrum R we will say a filtered R-module spectrum for an object in $\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$.

2. The phrase "spectral stack", will always refer to a stack for the π_*-flat topology on $\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{op}}$ as defined in Definition III.2.2.24.

3. For a Dirac stack X, $\mathrm{QCoh}(X)$ will always denote the stable ∞-category of quasi-coherent sheaves on X, as opposed to the abelian category defined for instance via the heart of the canonical t-structure on $\mathrm{QCoh}(X)$.

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Chapter III.2

Filtered stacks: Affine schemes

III.2.1 Filtered spectra

In this section we recall the fundamentals of filtered spectra, and establish notation which will be used in the rest of the paper.

Definition III.2.1.1. A *filtered spectrum* is a functor $X\colon \mathbb{Z}^{\mathrm{op}} \to \mathcal{S}p$. Here, $\mathbb{Z}$ is the integers considered as a poset with the usual ordering. We denote the category of filtered spectra by $\mathcal{S}p^{\mathrm{fil}} := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{S}p)$. The *underlying spectrum* of a filtered spectrum is given by the colimit of the diagram $X\colon \mathbb{Z}^{\mathrm{op}} \to \mathcal{S}p$. A *graded spectrum* is a functor $Y\colon \mathbb{Z}_{\delta}^{\mathrm{op}} \to \mathcal{S}p$. Here, $\mathbb{Z}_{\delta}$ is the category with objects $\mathbb{Z}$ and no non-identity morphisms. We denote the category of graded spectra by $\mathcal{S}p^{\mathrm{gr}} := \mathrm{Fun}(\mathbb{Z}_{\delta}^{\mathrm{op}}, \mathcal{S}p)$.

Remark III.2.1.2. We do not usually denote the filtration degree in our notation for a filtered spectrum. We justify this decision because of the following fact: the constant functor $\mathrm{const}\colon \mathcal{S}p \to \mathcal{S}p^{\mathrm{fil}}$ is fully faithful and hence any spectrum may trivially be considered a filtered spectrum by giving it the constant filtration.

Remark III.2.1.3. The categories $\mathbb{Z}$ and $\mathbb{Z}_\delta$ admit symmetric monoidal structures, induced by addition on $\mathbb{Z}$. As a consequence, $\mathcal{S}p^{\mathrm{fil}}$ and $\mathcal{S}p^{\mathrm{gr}}$ both admit symmetric monoidal structures given by Day convolution. We denote the monoidal unit by $\mathbb{S}^{0,0}$ and $\mathbb{S}^{0,0}_{\mathrm{gr}}$ respectively. By construction, the functor $\mathbb{Z} \xrightarrow{よ} \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\Sigma^\infty_+} \mathcal{S}p^{\mathrm{fil}}$ is symmetric monoidal (here よ denotes the Yoneda embedding and $\mathcal{S}$ is the category of spaces), and hence for every $k \in \mathbb{Z}$ the object $\mathbb{S}^{0,k} := \Sigma^\infty_+ よ(k)$ is tensor invertible. The same construction works for graded spectra.

Convention III.2.1.4. For the remainder of this paper by a *commutative filtered ring spectrum* we mean an object in the category of commutative algebra objects in filtered spectra.

Notation III.2.1.5. For a filtered spectrum X, we denote the value at $n \in \mathbb{Z}$ by X_n. Furthermore, we let $X_{-\infty} := \mathrm{colim}_{\mathbb{Z}^{\mathrm{op}}} X$. For $n, k \in \mathbb{Z}$ we denote by $\Sigma^{n,k} X := \mathbb{S}^{0,k} \otimes \Sigma^n X$. It is not difficult to see that $(\Sigma^{n,k} X)_i = \Sigma^n X_{i+k}$. We use the same notation for graded spectra.

Remark III.2.1.6. For every filtered spectrum X we have that

$$\begin{aligned} \mathrm{map}_{\mathcal{S}p^{\mathrm{fil}}}(\mathbb{S}^{n,k}, X) &\simeq \Sigma^{-n} \mathrm{map}_{\mathcal{S}p^{\mathrm{fil}}}(\mathbb{S}^{0,k}, X) \\ &\simeq \Sigma^{-n} \mathrm{map}_{\mathcal{S}p^{\mathrm{fil}}}(\mathbb{S}^{0,0}, \Sigma^{0,-k} X) \\ &\simeq \Sigma^{-n} \mathrm{map}_{\mathcal{S}p}(\mathbb{S}, X_k) \\ &\simeq \Sigma^{-n} X_k. \end{aligned}$$

In particular, on π_0 we get that $\pi_0 \mathrm{map}_{\mathcal{S}p^{\mathrm{fil}}}(\mathbb{S}^{n,k}, X) \cong \pi_n X_k$.

Definition III.2.1.7. Let X be a filtered spectrum and $n, k \in \mathbb{Z}$, then we define the (n,k)-th homotopy group of X to be $\pi_{n,k}(X) := \pi_0 \mathrm{map}_{\mathcal{S}p^{\mathrm{fil}}}(\mathbb{S}^{n,k}, X) \cong \pi_n X_k$.

Remark III.2.1.8. The category of graded spectra is canonically equivalent to the product of categories $\prod_{\mathbb{Z}} \mathcal{S}p$. This equivalence is *not* symmetric monoidal. The monoidal structure on $\prod_{\mathbb{Z}} \mathcal{S}p$ is pointwise, on the other side the monoidal structure on $\mathcal{S}p^{\mathrm{gr}}$ induced by Day convolution is given by

$$(X \otimes Y)_k = \bigoplus_{a+b=k} X_a \otimes Y_b,$$

in grading k. In particular, the equivalence above is not even unital, since the monoidal unit for the pointwise symmetric monoidal structure is the constant graded spectrum, with the sphere spectrum in every degree.

Definition III.2.1.9. If $X \in \mathcal{S}p^{\mathrm{fil}}$ is a filtered spectrum, then the *associated graded* is the graded spectrum $\mathrm{gr}_* X$ with $\mathrm{gr}_n X \coloneqq \mathrm{cof}(X_{n+1} \to X_n)$.

Proposition III.2.1.10 ([Lur15, Prop. 3.2.1])**.** *The association $X \mapsto \mathrm{gr}_* X$ refines to a symmetric monoidal colimit-preserving functor $\mathrm{gr}_* \colon \mathcal{S}p^{\mathrm{fil}} \to \mathcal{S}p^{\mathrm{gr}}$. Furthermore, there is a canonical equivalence $\mathrm{gr}_* \mathbb{S}^{n,k} \simeq \mathbb{S}^{n,k}_{\mathrm{gr}}$.*

Remark III.2.1.11. The associated graded functor admits a right adjoint $F \colon \mathcal{S}p^{\mathrm{gr}} \to \mathcal{S}p^{\mathrm{fil}}$, which on objects is given by sending a graded spectrum X to the filtered spectrum

$$\dots \xrightarrow{0} X_n \xrightarrow{0} X_{n-1} \xrightarrow{0} \dots .$$

In fact, this adjunction is comonadic [Mou21, Prop. 4.2], so we obtain an equivalence

$$\mathcal{S}p^{\mathrm{gr}} \simeq \mathrm{Mod}_{F(\mathbb{S}^{0,0}_{\mathrm{gr}})}(\mathcal{S}p^{\mathrm{fil}})$$

One can check that $F(\mathbb{S}^{0,0}_{\mathrm{gr}})$ is the filtered spectrum

$$\cdots \to 0 \to \mathbb{S} \to 0 \to \dots ,$$

where the sphere sits in filtration 0. We will denote this filtered spectrum by $C\tau \coloneqq F(\mathbb{S}^{0,0}_{\mathrm{gr}})$. Since the associated graded functor is symmetric monoidal this provides a canonical commutative algebra structure on $C\tau$. In fact, for every filtered spectrum $X \in \mathcal{S}p^{\mathrm{fil}}$ there is a map $\tau\colon \Sigma^{0,-1}X \to X$, where the n-th component is given by $X_{n+1} \to X_n$. The filtered spectrum $C\tau$ can be canonically identified with $\mathrm{cof}(\mathbb{S}^{0,-1} \xrightarrow{\tau} \mathbb{S}^{0,0})$. C.f [BHS26, Appendix B] or [Lur15, Section 3.2.].

Definition III.2.1.12. A filtered spectrum $X \in \mathcal{S}p^{\mathrm{fil}}$ is *complete* if the limit $\lim_{\mathbb{Z}^{\mathrm{op}}} X \simeq 0$.

Remark III.2.1.13. The full subcategory of complete filtered spectra is closed under limits in filtered spectra. In fact, it is the Bousfield localization at $C\tau$, and thus $\mathrm{gr}_*\colon \mathcal{S}p^{\mathrm{fil}} \to \mathcal{S}p^{\mathrm{gr}}$ is conservative on the full subcategory of complete spectra. This can be verified as follows. If $\mathrm{gr}_* X \simeq 0$, then $X_n \to X_{n-1}$ is an equivalence, for all $n \in \mathbb{Z}$. In particular, if X is complete, then $X \simeq 0$.

Proposition III.2.1.14. *If X is a filtered spectrum, then there is a spectral sequence with signature*

$$E_1^{p,q} = \pi_{p,q}(\mathrm{gr}_* X) \Rightarrow \pi_p(\mathrm{colim}_{k\to -\infty} X_k),$$

such that the differential d_r has bidegree $(-1, r)$. Furthermore, if X is complete, then the spectral sequence is conditionally convergent [Nig25, Definition 2.46].

Definition III.2.1.15. If X is a filtered spectrum, then the *associated spectral sequence* is the spectral sequence constructed in Proposition III.2.1.14.

Remark III.2.1.16. The spectral sequence associated to a filtered spectrum X is constructed via an exact couple. This exact couple is induced by the cofiber sequence $\Sigma^{0,-1}X \xrightarrow{\tau} X \to C\tau \otimes X$. In [Lur17b, Section 1.2.2] Lurie defines an a priori different spectral sequence associated to a filtered spectrum. This is done by writing the Cartan–Eilenberg system in the language of stable ∞-categories. The two spectral sequences have been compared by [Cre25, Theorem 5.1], who shows that they are indeed the same.

Remark III.2.1.17. Many spectral sequences of interest start at the E_2-page. One can obtain a *second-page reindexing* via

$$'E_2^{p,q} = E_1^{p,p+q}.$$

We refer the reader to [Nig25, Remark 3.27] for a more in depth explanation.

Remark III.2.1.18. There is a well-established theory for constructing a spectral sequence from a filtered object in stable ∞-categories equipped with a right complete t-structure which is compatible with sequential colimits. This will be explored further in Appendix A.1.

We will now introduce a large class of complete filtered spectra.

Definition III.2.1.19. We say that a filtered spectrum X is *connective* if X_n is n-connective for every $n \in \mathbb{Z}$.

Proposition III.2.1.20. *The full subcategory of connective filtered spectra forms the connective part of a t-structure on $\mathrm{Sp}^{\mathrm{fil}}$. This t-structure satisfies the following:*

1. *The t-structure is compatible with the symmetric monoidal structure on $\mathrm{Sp}^{\mathrm{fil}}$, and the t-structure is compatible with filtered colimits.*

2. *The t-structure is left and right complete.*

3. *The heart is equivalent to graded abelian groups, and via this identification the functor $\pi_0^P : \mathcal{S}p^{\mathrm{fil}} \to \mathrm{Ab}^{\mathrm{gr}}$ can be identified with the functor sending a filtered spectrum X to the graded abelian group $\{\pi_k X_k\}_{k\in\mathbb{Z}}$.*

Furthermore, one may define a t-structure on graded spectra, with the same properties.

Proof. Everything except for (2) can be found in [Hed20, Prop. II.1.22, Prop. II.1.23, Prop. II.1.24]. To see that (2) holds, it suffices by [Lur17b, Prop .1.2.1.19] - and its dual version - to see that any infinitely (co)connective object is 0, which follows immediately from the fact that the standard t-structure on $\mathcal{S}p$ is left and right complete. Note, that the conditions of [Lur17b, Prop. 1.2.1.19] follow immediately from the fact that they are satisfied by the standard t-structure on $\mathcal{S}p$ and that being (co)connective is a pointwise condition in $\mathcal{S}p^{\mathrm{fil}}$. $\square$

Definition III.2.1.21. We say that a filtered spectrum $X \in \mathcal{S}p^{\mathrm{fil}}$ is *connective* if it is connective in the t-structure constructed in Proposition III.2.1.20.

Remark III.2.1.22. There are many t-structures on filtered spectra, which serve many different purposes. In this paper we will only concern ourselves with the above t-structure. This t-structure usually goes under the names the diagonal, Postnikov, or standard t-structure on filtered spectra.

Proposition III.2.1.23. *The Postnikov t-structure on $\mathcal{S}p^{\mathrm{fil}}$ is accessible, that is, the subcategory $\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0}$ is presentable. In fact, it is the t-structure generated by the collection $\{\mathbb{S}^{n,n}\}_{n\in\mathbb{Z}}$ in the sense of [Lur17b, Theorem 1.4.4.11].*

Proof. We first verify that $\mathbb{S}^{n,n}$ is connective for all $n \in \mathbb{Z}$. We have to show that $\pi_{a,b}\mathbb{S}^{n,n} \cong 0$, whenever $a < b$. To see this we compute:

$$\begin{aligned}\pi_{a,b}\mathbb{S}^{n,n} &\cong \pi_{a-n,b-n}\mathbb{S}^{0,0} \\ &\cong \begin{cases}\pi_{a-n}\mathbb{S} & b \leqslant n \\ 0 & \text{else.}\end{cases}\end{aligned}$$

If $a < b$, then $a - n < 0$. In particular, $\pi_{a-n}\mathbb{S} \cong 0$, since the sphere spectrum is connective.

Now, let $\mathcal{C}$ be the smallest subcategory of $\mathcal{S}p^{\mathrm{fil}}$ containing $\{\mathbb{S}^{n,n}\}_{n\in\mathbb{Z}}$ which is closed under extensions and colimits. It is clear from the above that $\mathcal{C} \subseteq \mathcal{S}p^{\mathrm{fil}}_{\geqslant 0}$, so it suffices to see that for every $X \in \mathcal{S}p^{\mathrm{fil}}_{\geqslant 0}$ we have that $X \in \mathcal{C}$. In order to do so we follow the proof of [Lur17b, Prop. 7.1.1.13]. Let $X(0) \coloneqq \bigoplus_{n\in\mathbb{Z}} \bigoplus_{\alpha\in\pi_{n,n}X} \mathbb{S}^{n,n}$ and let $f_0\colon X(0) \to X$ denote the induced map. It follows from the long exact sequence in homotopy groups that $F(0) \coloneqq \mathrm{fib}(f_0)$ is connective since $\pi_0^P f_0$ is surjective. Let us now define $D(0) \coloneqq \bigoplus_{n\in\mathbb{Z}} \bigoplus_{\alpha\in\pi_{n,n}F(0)} \mathbb{S}^{n,n}$ and let $d_0\colon D(0) \to F(0)$ be the induced map. To finish the proof we now inductively define $X(i+1)$ to be the pushout

$$\begin{tikzcd} D(i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\ X(i) \arrow[r] & X(i+1). \end{tikzcd}$$

One sees inductively, that $X(0) \to X$ extends to a map $f_i\colon X(i) \to X$ and we define $F(i) \coloneqq \mathrm{fib}(f_i)$, $D(i) \coloneqq \bigoplus_{n\in\mathbb{Z}} \bigoplus_{\alpha\in\pi_{n+i,n}F(i)} \mathbb{S}^{n+i,n}$ and d_i is the canonical map $D(i) \to F(i)$. It can be checked that $F(i)$ is i-connective for all i and thus $\mathrm{colim}_{i\to\infty} X(i) \to X$ is an equivalence, which is what we wanted to prove. $\square$

Remark III.2.1.24. If X is a filtered spectrum, then the t-structure homotopy groups of X are given by

$$\pi_k^P X \cong \{\pi_{n+k,n}X\}_{n\in\mathbb{Z}} \cong \{\pi_{n+k}X_n\}_{n\in\mathbb{Z}}.$$

Lemma III.2.1.25. *If R is a connective filtered ring spectrum and M, N are connective filtered R-module spectra, then the relative tensor product $M \otimes_R N$ is connective.*

Proof. One can compute the relative tensor product as a bar construction:

$$M \underset{R}{\otimes} N \simeq |M \otimes R^{\otimes \bullet} N|.$$

Since the t-structure on filtered spectra is compatible with the symmetric monoidal structure, $M \otimes R^k \otimes N$ is connective for every $k \geqslant 0$. Connective objects are closed under colimits in filtered spectra, so the conclusion follows. $\square$

If R is a connective filtered ring spectrum, then $\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$ has a natural t-structure such that the forgetful functor is t-exact, that is, it is compatible with the one produced in Proposition III.2.1.20. This is not strictly necessary for any of the arguments we make, but it makes certain statements needed later more clear.

Proposition III.2.1.26. *Let R be a filtered ring spectrum and let $\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0}$ and $\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})_{\leqslant 0}$ denote the preimage of $\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0}$ and $\mathcal{S}p^{\mathrm{fil}}_{\leqslant 0}$ respectively. If R is connective, then this determines a t-structure on $\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$ such that the following hold:*

(0) The forgetful functor $\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}}) \to \mathcal{S}p^{\mathrm{fil}}$ is t-exact.

1. *The t-structure is compatible with the symmetric monoidal structure on $\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$, and is compatible with filtered colimits.*
2. *The t-structure is left and right complete.*
3. *The heart is equivalent to $\mathrm{Mod}_{\pi_0^P R}(\mathrm{Ab}^{\mathrm{gr}})$, and via this identification the functor $\pi_0^P \colon \mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}}) \to \mathrm{Mod}_{\pi_0^P R}(\mathrm{Ab}^{\mathrm{gr}})$ can be identified with the functor sending a filtered spectrum X to the $\pi_0^P R$-module $\{\pi_k X_k\}_{k \in \mathbb{Z}}$.*

Proof. This is the special case of Proposition A.1.11 where $\mathcal{C} = \mathcal{S}p^{\mathrm{fil}}$. $\square$

One reason why one might care about connective filtered spectra is the following result.

Proposition III.2.1.27. *If a filtered spectrum is connective, then it is complete.*

Proof. Let X be a connective filtered spectrum. We have to show that $\lim_{\mathbb{Z}^{\mathrm{op}}} X \simeq 0$. Now consider the Milnor $\lim^1$-sequence

$$0 \to \lim_i{}^1 \pi_{k+1}(X_i) \to \pi_k(\lim_i X_i) \to \lim_i \pi_k(X_i) \to 0.$$

Since X is assumed connective we have that $\pi_k(X_i) \cong 0$ for $i > k$. In particular, $\lim_i \pi_k(X_i) \simeq 0$ for every k and the Mittag–Leffler condition is satisfied. $\square$

Construction III.2.1.28. Let $\mathbb{S}^{1,1}_{\mathrm{gr}}$ denote the graded $(1,1)$-sphere; that is the graded spectrum with $\mathbb{S}^1$ in degree 1 and which is 0 in every other degree. We define $\mathbb{S}\{\mathbb{S}^{1,1}_{\mathrm{gr}}\} \coloneqq \mathrm{Free}_{\mathbb{E}_1}(\mathbb{S}^{1,1}_{\mathrm{gr}})$ to be the free associative algebra on $\mathbb{S}^{1,1}_{\mathrm{gr}}$ in graded spectra. It follows from [Lur17b, Prop. 4.1.1.18] that the underlying graded spectrum of $\mathbb{S}\{\mathbb{S}^{1,1}_{\mathrm{gr}}\}$ is given by

$$\mathbb{S}\{\mathbb{S}^{1,1}_{\mathrm{gr}}\} \simeq \bigoplus_{i \in \mathbb{Z}_{\geqslant 0}} (\mathbb{S}^{1,1}_{\mathrm{gr}})^{\otimes i} \simeq \bigoplus_{i \in \mathbb{Z}_{\geqslant 0}} \mathbb{S}^{i,i}_{\mathrm{gr}}.$$

Precomposing with the Yoneda embedding defines an $\mathbb{E}_1$-monoidal functor

$$\iota \colon \mathbb{Z}_\delta \to \mathrm{Pic}(\mathcal{S}p)$$

sending 1 to $\mathbb{S}^1$. Hence, we obtain a lax monoidal functor

$$\mathbb{Z}_\delta \times \mathcal{S}p^{\mathrm{gr}} \xrightarrow{(\iota, \mathrm{ev}_0)} \mathrm{Pic}(\mathcal{S}p) \times \mathcal{S}p \xrightarrow{\otimes} \mathcal{S}p$$

which defines a lax monoidal functor

$$\Sigma^* \colon \mathcal{S}p^{\mathrm{gr}} \to \mathcal{S}p^{\mathrm{gr}},$$

given on objects by $\{X_k\}_{k\in\mathbb{Z}} \mapsto \{\Sigma^k X_k\}_{k\in\mathbb{Z}}$. It is easy to verify that this lax monoidal functor is monoidal. We name this functor the *shearing functor*.

Remark III.2.1.29. There is also a *double shearing functor* given by $\{X_k\}_{k\in\mathbb{Z}} \mapsto \{\Sigma^{2k} X_k\}_{k\in\mathbb{Z}}$. This can be seen to be symmetric monoidal when working $\mathbb{Z}$-linearly or even MU-linearly. See for example [Dev24, Lemma 2.1.3].

Remark III.2.1.30. If $\tilde{H} \colon \mathrm{Ab}^{\mathrm{gr}} \to \mathcal{S}p^{\mathrm{gr}}$ denotes the inclusion of the heart, and $M \coloneqq \{M_k\}_{k\in\mathbb{Z}}$ is a graded abelian group, then the graded spectrum $\tilde{H}(M)$ is given by $\{\Sigma^k H(M_k)\}_{k\in\mathbb{Z}}$, where $H \colon \mathrm{Ab} \to \mathcal{S}p$ is the usual Eilenberg–Maclane functor. This also agrees with the composite of the shearing functor Σ^* and applying H level-wise. We will usually not denote the functors $\tilde{H}$ and H.

Definition III.2.1.31. For a spectrum X the *Whitehead tower* of X is the filtered spectrum

$$\tau_{\geqslant\bullet} X \coloneqq [\cdots \to \tau_{\geqslant n} X \to \tau_{\geqslant n-1} X \to \dots].$$

Here $\tau_{\geqslant n} X$ sits in filtration degree n.

Remark III.2.1.32. For any spectrum X, the Whitehead tower $\tau_{\geqslant\bullet} X$ is a connective filtered spectrum.

Proposition III.2.1.33 ([Hed20, Prop. II.1.26])**.** *The association $X \mapsto \tau_{\geqslant\bullet} X$ refines to a lax symmetric monoidal functor $\tau_{\geqslant\bullet} \colon \mathcal{S}p \to \mathcal{S}p^{\mathrm{fil}}$.*

Proof. The functor can be defined as the composite

$$\mathcal{S}p \xrightarrow{\mathrm{const}} \mathcal{S}p^{\mathrm{fil}} \xrightarrow{\tau^P_{\geqslant 0}} \mathcal{S}p^{\mathrm{fil}}_{\geqslant 0} \hookrightarrow \mathcal{S}p^{\mathrm{fil}},$$

all of which are lax symmetric monoidal. □

Example III.2.1.34. For a spectrum X the associated graded of the Whitehead tower of X is the graded spectrum $\Sigma^*\pi_*X$ whose k-th graded piece is given by $\Sigma^k\pi_kX$. In particular, it is discrete in the diagonal t-structure on $\mathcal{S}p^{\mathrm{gr}}$.

Example III.2.1.35. It is often possible to write spectral sequences of interest as the spectral sequence associated to a filtered spectrum. We learned the following list of examples from Robert Burklund.

1. If G is a finite group and X is a spectrum with a G-action, then the spectral sequence associated to the filtered spectrum
$$\tau_{\geqslant\bullet}(X)^{hG}$$
is a version of the homotopy fixed points spectral sequence for X and G. To see this note that the E_1-page is given by the bigraded homotopy groups of
$$\mathrm{gr}_*(\tau_{\geqslant\bullet}(X)^{hG}) \simeq (\Sigma^*\pi_*X)^{hG}.$$
2. If X is a space and M is any spectrum, then the spectral sequence associated to $\tau_{\geqslant\bullet}M \otimes \Sigma^\infty_+ X$ is a version of the Atiyah–Hirzebruch spectral sequence computing $M_*(X)$.

Example III.2.1.36. For every $X \in \mathcal{S}p$ we can compute the t-structure homotopy groups of $\tau_{\geqslant\bullet}X$ as
$$\pi_k^P\tau_{\geqslant\bullet}X \cong \{\pi_{n+k}\tau_{\geqslant n}X\}_{n\in\mathbb{Z}} \cong \begin{cases}\{\pi_{n+k}X\}_{n\in\mathbb{Z}} & k \geqslant 0\\ 0 & k<0.\end{cases}$$
In fact, for any connective filtered spectrum Y, the map $Y \to \mathrm{gr}_*Y$, induced by the unit map $\mathbb{S}^{0,0} \to C\tau$, is the 0-truncation of Y.

Construction III.2.1.37. If $f\colon R \to S$ is a map of ring spectra, then there is a commutative square:

$$\begin{tikzcd}
\mathrm{Mod}_S \arrow[r, "\tau_{\geqslant\bullet}"] \arrow[d, "f_*"] & \mathrm{Mod}_{\tau_{\geqslant\bullet}S}(\mathcal{S}p^{\mathrm{fil}}) \arrow[d, "(\tau_{\geqslant\bullet}f)_*"] \\
\mathrm{Mod}_R \arrow[r, "\tau_{\geqslant\bullet}"] & \mathrm{Mod}_{\tau_{\geqslant\bullet}R}(\mathcal{S}p^{\mathrm{fil}}).
\end{tikzcd}$$

Here the vertical functors are restriction of scalars. Now, since both f_* and $(\tau_{\geqslant\bullet})_*$ admit left adjoints, we obtain a canonical Beck–Chevalley natural transformation

$$\phi_f\colon (\tau_{\geqslant\bullet}f)^*\tau_{\geqslant\bullet} \to \tau_{\geqslant\bullet}(f^*),$$

which on an R-module M produces a map

$$\phi_{f,M}\colon \tau_{\geqslant\bullet}S \underset{\tau_{\geqslant\bullet}R}{\otimes} \tau_{\geqslant\bullet}M \to \tau_{\geqslant\bullet}(S \underset{R}{\otimes} M).$$

In the next section we will study flat maps of filtered ring spectra. One particular example of such maps are the maps induced by π_*-flat maps via the Postnikov tower, in this situation the map ϕ from Construction III.2.1.37 is an equivalence.

III.2.2 The flat topology on filtered ring spectra

We will now define and study flat maps of filtered ring spectra. In order to do so, we use the theory of Dirac geometry developed by Hesselholt and Pstrągowski in [HP25] and [HP24].

Definition III.2.2.1 ([HP25, Definition 3.6])**.** Let $R \in \mathrm{CAlg}(\mathrm{Ab}^{\mathrm{gr}})$ be a Dirac ring, then an R-module M is *flat* if the functor

$$M \underset{R}{\otimes} -\colon\ \mathrm{Mod}_R(\mathrm{Ab}^{\mathrm{gr}}) \to \mathrm{Mod}_R(\mathrm{Ab}^{\mathrm{gr}})$$

is exact.

Remark III.2.2.2. Recall that the t-structure homotopy groups on filtered spectra[1] define functors $\pi_k^P \colon \mathcal{S}p^{\mathrm{fil}} \to \mathrm{Ab}^{\mathrm{gr}}$, for every $k \in \mathbb{Z}$. Recall that the Postnikov t-structure is compatible with the symmetric monoidal structure on $\mathcal{S}p^{\mathrm{fil}}$. In particular, if R is a filtered ring spectrum, then $\pi_0^P R$ is a Dirac ring, and $\pi_n^P R$ is a graded module over $\pi_0^P R$. Furthermore, if M is a graded R-module spectrum, then $\pi_k^P M$ is a $\pi_0^P R$-module for every $k \in \mathbb{Z}$.

Definition III.2.2.3. Let $R \in \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})$ be a filtered ring spectrum. A filtered R-module spectrum $M \in \mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$ is *flat* if $\pi_0^P M$ is flat as a graded module over the Dirac ring $\pi_0^P R$ and the canonical map

$$\pi_n^P R \underset{\pi_0^P R}{\otimes} \pi_0^P M \to \pi_n^P M$$

is an equivalence. A map of filtered ring spectra $R \to S$ is *flat* if S is flat when considered as an R-module.

Lemma III.2.2.4. *Let $R \to S$ be a map of filtered ring spectra and let M be a filtered S-module spectrum. If M is flat over S and S is flat over R, then M is flat over R.*

Proof. It is clear that $\pi_0^P M$ is flat over $\pi_0^P R$. So it remains to see that

$$\pi_n^P R \underset{\pi_0^P R}{\otimes} \pi_0^P M \to \pi_n^P M$$

is an equivalence for every n. Since M is a flat S-module and $\pi_0^P M$ is a $\pi_0^P S$-module, we see that every map in the composite

$$\begin{tikzcd}
\pi_n^P R \otimes_{\pi_0^P R} \pi_0^P M \arrow[r, "\sim"] & \pi_n^P R \otimes_{\pi_0^P R} \pi_0^P S \otimes_{\pi_0^P S} \pi_0^P M \arrow[d, "\sim"] & \\
& \pi_n^P S \otimes_{\pi_0^P S} \pi_0^P M \arrow[r, "\sim"] & \pi_n^P M
\end{tikzcd}$$

[1] With respect to the Postnikov t-structure.

is an equivalence. Hence, so is the composite, which is what we wanted to prove. □

Remark III.2.2.5. This is analogous to the definition of flatness introduced by Lurie in [Lur17b]. It is possible to make an analogous definition in every symmetric monoidal stable ∞-category with a t-structure which is compatible with the symmetric monoidal structure. We develop this idea in Appendix A.1.

Proposition III.2.2.6. *If R is a filtered ring spectrum and M, N are filtered R-module spectra, then there is a spectral sequence with signature*

$$E_1^{p,q} = \pi_{p,q}(\Sigma^* \pi_*^P M \underset{\Sigma^* \pi_*^P R}{\otimes} \Sigma^* \pi_*^P N) \Rightarrow \pi_p^P (M \underset{R}{\otimes} N).$$

Furthermore, if R, M and N are bounded below, then the spectral sequence is convergent.

Proof. This is an application of Proposition A.1.4. □

Remark III.2.2.7. Recall that the functor Σ^* is monoidal, so if R, M and N are as in Proposition III.2.2.6 we obtain an equivalence

$$\Sigma^* \pi_*^P M \underset{\Sigma^* \pi_*^P R}{\otimes} \Sigma^* \pi_*^P N \simeq \Sigma^* (\pi_*^P M \underset{\pi_*^P R}{\otimes} \pi_*^P N).$$

Definition III.2.2.8. Let R, M and N be as above, then define the *Künneth spectral sequence* computing $\pi_*^P(M \otimes_R N)$ to be the spectral sequence in Proposition III.2.2.6.

Proposition III.2.2.9. *Let R be a connective filtered ring spectrum and M, N be connective filtered R-module spectra. If M is flat, then there is an isomorphism*

$$\pi_*^P (M \underset{R}{\otimes} N) \cong \pi_*^P M \underset{\pi_*^P R}{\otimes} \pi_*^P N \cong \pi_0^P M \underset{\pi_0^P R}{\otimes} \pi_*^P N.$$

Proof. This is Proposition A.1.24 applied to the situation above. $\square$

Definition III.2.2.10. If R is a filtered ring spectrum and M is a filtered R-module spectrum, then we say that M is *faithful* if the functor

$$M \underset{R}{\otimes} - \colon \operatorname{Mod}_R(\mathcal{S}p^{\mathrm{fil}}) \to \operatorname{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$$

is conservative. We say that a map of filtered ring spectra $R \to S$ is *faithfully flat* if and only if S is faithful and flat as an R-module.

Remark III.2.2.11. If $f \colon R \to S$ is a flat map of connective filtered ring spectra, then f is faithful if and only if $\pi_0^P f \colon \pi_0^P R \to \pi_0^P S$ is faithful. This is an easy application of the Künneth spectral sequence.

Proposition III.2.2.12. *A map of connective filtered ring spectra $f \colon R \to S$, which is a monomorphism on π_*, is faithfully flat if and only if* $\operatorname{cof} f$ *is flat as an R-module.*

Proof. Suppose first that f is faithfully flat. It follows from Remark III.2.2.11 and [HP25, Addendum 3.9] that $\pi_0 \operatorname{cof} f$ is flat. So it remains to see that the canonical map

$$\pi_n^P R \underset{\pi_0^P R}{\otimes} \pi_0 \operatorname{cof} f \to \pi_n^P \operatorname{cof} f$$

is an equivalence. This follows from the 5-lemma and using the long exact sequence in homotopy for the cofiber sequence $R \to S \to \operatorname{cof} f$.

Suppose instead that $\operatorname{cof} f$ is flat, then using [HP25, Addendum 3.9] again we obtain that $\pi_0^P R \to \pi_0^P S$ is faithfully flat. It is easy to see the base-change condition is verified using the long exact sequence in homotopy. $\square$

Proposition III.2.2.13. *Let R be a connective filtered ring spectrum. If M is a flat R-module, then the functor*

$$M \underset{R}{\otimes} - \colon \operatorname{Mod}_R(\mathcal{S}p^{\mathrm{fil}}) \to \operatorname{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$$

is t-exact.

Proof. Since R is connective and M is flat, it follows that M is also connective. Hence $M \otimes_R -$ is right t-exact. Thus it suffices to show that if N is a discrete R-module, then so is $M \otimes_R N$. To see this we note that by Corollary A.1.25 we have equivalences

$$\pi_*^P(M \underset{R}{\otimes} N) \cong \pi_*^P(M) \underset{\pi_*^P R}{\otimes} \pi_*^P N \cong \pi_0^P M \underset{\pi_0^P R}{\otimes} \pi_*^P N$$

in graded abelian groups. The result follows immediately from our assumption that $\pi_k^P N \cong 0$ for $k \neq 0$. $\square$

Definition III.2.2.14. A finite collection of maps of filtered ring spectra $\{R \to S_i\}_{i \in I}$ is called a *flat cover* if the map

$$R \to \prod_{i \in I} S_i$$

is faithfully flat.

Proposition III.2.2.15. *If R is a connective filtered ring spectrum, then there exists a finitary Grothendieck topology on $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})^{\mathrm{op}}_R$. If $A \in \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})_R$, then a sieve $\mathcal{C} \subseteq (\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})^{\mathrm{op}}_R)_{/A}$ is a covering sieve if and only if it contains a flat cover of A.*

Proof. The proof in [Lurb, Prop. B.6.1.3] carries over verbatim. The only technical input is Proposition III.2.2.9. $\square$

Definition III.2.2.16. Let R be a connective filtered ring spectrum. We refer to the Grothendieck topology in Proposition III.2.2.15 as the *flat topology* on $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})^{\mathrm{op}}_R$. We denote the site given by $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})^{\mathrm{op}}$ with the flat topology by FAff.

Proposition III.2.2.17. *Let R be a connective filtered ring spectrum and $\mathcal{C}$ be a category with all limits. A functor $F\colon \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})_R \to \mathcal{C}$ is a sheaf for the flat topology if and only if the following conditions are satisfied.*

1. *F preserves finite products.*
2. *For every flat cover $R \to S$ the canonical map*
$$F(R) \to \lim_{[n]\in\Delta} F(S^{\otimes_R n+1})$$
is an equivalence.

Proof. This follows immediately from [Lurb, Prop. A.3.3.1]. $\square$

Remark III.2.2.18. There is a version of Proposition III.2.2.15 and Proposition III.2.2.17 where we do not require connectivity. This however is not relevant for this paper, since the objects of interest to us are usually connective.

Definition III.2.2.19. Let R be a ring spectrum. An R-module spectrum M is *π_*-flat* over R if and only if $\pi_* M$ is flat over the Dirac ring $\pi_* R$.

Lemma III.2.2.20. *If $f\colon R \to S$ is a π_*-flat map of ring spectra, then for any R-module spectrum M the map*
$$\phi_{f,M}\colon \tau_{\geqslant\bullet} S \underset{\tau_{\geqslant\bullet} R}{\otimes} \tau_{\geqslant\bullet} M \to \tau_{\geqslant\bullet}(S \underset{R}{\otimes} M)$$
defined in Construction III.2.1.37 is an equivalence.

Proof. By Lemma III.2.1.25 the left-hand side is connective, hence both sides are. It follows from Proposition III.2.1.27 that both filtered spectra are complete, so it suffices to show that the induced map on associated graded is an equivalence. Now, $\mathrm{gr}_*\colon \mathcal{S}p^{\mathrm{fil}} \to \mathcal{S}p^{\mathrm{gr}}$ is symmetric monoidal so we have an equivalence
$$\begin{aligned}\mathrm{gr}_*(\tau_{\geqslant\bullet} S \underset{\tau_{\geqslant\bullet} R}{\otimes} \tau_{\geqslant\bullet} M) &\simeq \mathrm{gr}_* \tau_{\geqslant\bullet} S \underset{\mathrm{gr}_* \tau_{\geqslant\bullet} R}{\otimes} \mathrm{gr}_* \tau_{\geqslant\bullet} M \\ &\simeq \Sigma^* \pi_* S \underset{\Sigma^* \pi_* R}{\otimes} \Sigma^* \pi_* M.\end{aligned}$$

Hence, we may reduce to showing that the map

$$\Sigma^*\pi_*S \underset{\Sigma^*\pi_*R}{\otimes} \Sigma^*\pi_*M \to \Sigma^*\pi_*(S \underset{R}{\otimes} M)$$

is an equivalence. One can see that the above map is an equivalence by following the proof of [Lur17b, Prop. 7.2.2.13]. The key input is that since S is π_*-flat over R, then the Dirac ring π_*S is flat over π_*R, so the derived tensor product has no non-trivial higher homotopy groups. □

Proposition III.2.2.21. *A map of ring spectra $f\colon R \to S$ is (faithfully) π_*-flat if and only if the map $\tau_{\geqslant\bullet}f\colon \tau_{\geqslant\bullet}R \to \tau_{\geqslant\bullet}S$ is (faithfully) flat.*

Proof. Suppose $f\colon R \to S$ is π_*-flat. We know that $\pi_0^P(\tau_{\geqslant\bullet}(-)) \cong \pi_*(-)$, so we immediately conclude that $\pi_0^P(\tau_{\geqslant\bullet}S)$ is flat over $\pi_0^P(\tau_{\geqslant\bullet}R)$ if and only if π_*S is flat over π_*R. It remains to check that if f is π_*-flat, then the base-change condition holds. This follows immediately from Example III.2.1.36.

Suppose now that $f\colon R \to S$ is faithfully flat. We have to check that $\tau_{\geqslant\bullet}f$ is faithful. By Remark III.2.2.11 this can be checked on π_0, where it follows from $f\colon R \to S$ being faithful.

Finally, suppose that $\tau_{\geqslant\bullet}f$ is faithfully flat. If M is an R-module with $S \otimes_R M \simeq 0$, then by Lemma III.2.2.20 we have that

$$\tau_{\geqslant\bullet}S \underset{\tau_{\geqslant\bullet}R}{\otimes} \tau_{\geqslant\bullet}M \simeq \tau_{\geqslant\bullet}(S \underset{R}{\otimes} M) \simeq 0.$$

Now, since $\tau_{\geqslant\bullet}f$ is faithfully flat we see that $\tau_{\geqslant\bullet}M \simeq 0$, hence $M \simeq 0$. □

Corollary III.2.2.22. *The functor $\tau_{\geqslant\bullet}\colon \mathrm{CAlg} \to \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})$ takes π_*-flat covers to flat covers.*

Corollary III.2.2.23. *The collection of jointly faithfully π_*-flat maps of commutative ring spectra forms a finitary Grothendieck topology on $\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{op}}$.*

Definition III.2.2.24. A finite collection $\{R \to S_i\}$ of π_*-flat maps of commutative ring spectra is a *π_*-flat cover* if they are jointly faithfully π_*-flat. This forms a Grothendieck topology by Corollary III.2.2.23 and we denote the site consisting of $\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{op}}$ with the π_*-flat topology by Aff.

Proposition III.2.2.25. *If R is a commutative ring spectrum, and $f\colon \tau_{\geqslant \bullet} R \to S'$ is a flat map of filtered commutative ring spectra, then there exists a π_*-flat map of commutative ring spectra $R \to S$ and an equivalence*

$$\tau_{\geqslant \bullet} S \simeq S'$$

of filtered commutative ring spectra.

Proof. Since S' is flat over $\tau_{\geqslant \bullet} R$, we see immediately that S' is connective. Furthermore, the map $\Sigma^* \pi_* R \to \mathrm{gr}_* S'$ is flat, since flat maps are closed under base-change. In particular, $\mathrm{gr}_* S'$ is discrete in the Postnikov t-structure. Since S' is connective, this already tells us that as a filtered spectrum $S' \simeq \tau_{\geqslant \bullet}(S'[\tau^{-1}])$. We will now promote this to an equivalence of commutative algebra objects in $\mathcal{S}p^{\mathrm{fil}}$. Since $\mathcal{S}p^{\mathrm{fil}} \xrightarrow{\tau^{-1}} \mathcal{S}p$ is symmetric monoidal, we get an induced adjunction

$$\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}) \underset{\mathrm{const}}{\overset{\tau^{-1}}{\rightleftarrows}} \mathrm{CAlg}(\mathcal{S}p).$$

In particular, we get an induced map $S' \to \mathrm{const}(S'[\tau^{-1}])$ of commutative algebras. Now, since S' is connective, this factors as a map of commutative algebras through the connective cover $\tau_{\geqslant \bullet}(S'[\tau^{-1}]) \to \mathrm{const}(S'[\tau^{-1}])$. The induced map $S' \to \tau_{\geqslant \bullet}(S'[\tau^{-1}])$ is an equivalence, by the above argument. $\square$

Corollary III.2.2.26. *For any commutative ring spectrum R, the functor*

$$\tau_{\geqslant \bullet}\colon\ \mathrm{CAlg}(\mathcal{S}p)_{R/} \to \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})_{\tau_{\geqslant \bullet} R/}$$

restricts to an equivalence

$$\tau_{\geqslant \bullet}\colon \operatorname{CAlg}(\mathcal{S}p)^{\flat}_{R/} \xrightarrow{\sim} \operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})^{\flat}_{\tau_{\geqslant \bullet}R/}.$$

Here $\operatorname{CAlg}(\mathcal{S}p)^{\flat}_{R/}$ *and* $\operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})^{\flat}_{\tau_{\geqslant \bullet}R/}$ *denote the full subcategories spanned by* π_**-flat commutative* R*-algebras and flat commutative* $\tau_{\geqslant \bullet}R$*-algebras respectively.*

Proof. The functor

$$\tau_{\geqslant \bullet}\colon \operatorname{CAlg}(\mathcal{S}p)_{R/} \to \operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})_{\tau_{\geqslant \bullet}R/}$$

is fully faithful, by a standard argument, see for instance Proposition III.3.3.11. Furthermore, the functor restricts to the respective flat subcategories by Proposition III.2.2.21 and is essentially surjective by Proposition III.2.2.25. □

Remark III.2.2.27. One might hope that $\pi_0^P\colon \operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0} \to \mathrm{DRing}$ exhibits similar behavior of determining flat algebras. However, this is false in general, a counter-example can be constructed using say group algebras for finite groups, cf. the introduction in[CNY24]. Instead, we consider one class of algebras where this is the case, namely the etale algebras. Note, that Hesselholt–Pstrągowski, show in [HP25, Theorem 4.33] that for a commutative ring spectrum R, there is an equivalence

$$\operatorname{CAlg}(\mathcal{S}p)^{\mathrm{et}}_{R/} \to \mathrm{DRing}^{\mathrm{et}}_{\pi_*R/}$$

between the full subcategory of $\operatorname{CAlg}(\mathcal{S}p)_{R/}$ and $\mathrm{DRing}_{\pi_*R/}$ spanned by π_*-etale commutative R-algebras and etale Dirac π_*R-algebras respectively.

Definition III.2.2.28. A map $R \to S$ of filtered commutative ring spectra is *etale* if it is flat and the induced map of Dirac rings $\pi_0^P R \to \pi_0^P S$ is etale in the sense of [HP25, Definition 4.7].

Remark III.2.2.29. It follows immediately that a map of commutative ring spectra $R \to S$ is π_*-etale if and only if $\tau_{\geqslant \bullet} R \to \tau_{\geqslant \bullet} S$ is an etale map of filtered commutative ring spectra.

Theorem III.2.2.30. If R is a commutative ring spectrum, then the functor
$$\pi_0^P \colon \operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})^{\mathrm{et}}_{\tau_{\geqslant \bullet} R/} \to \operatorname{DRing}^{\mathrm{et}}_{\pi_* R/}$$
is an equivalence. In particular, $\operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})^{\mathrm{et}}_{\tau_{\geqslant \bullet} R/}$ is a 1-category.

Proof. The functor
$$\pi_* \colon \operatorname{CAlg}(\mathcal{S}p)^{\mathrm{et}}_{R/} \to \operatorname{DRing}^{\mathrm{et}}_{\pi_* R/}$$
factors as
$$\begin{tikzcd} \operatorname{CAlg}(\mathcal{S}p)^{\mathrm{et}}_{R/} \arrow[rr, "\tau_{\geqslant \bullet}"] \arrow[dr, "\pi_*"'] & & \operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})^{\mathrm{et}}_{\tau_{\geqslant \bullet} R/} \arrow[dl, "\pi_0^P"] \\ & \operatorname{DRing}^{\mathrm{et}}_{\pi_* R/} & \end{tikzcd}$$
The functor $\tau_{\geqslant \bullet}$ is an equivalence by Corollary III.2.2.26 and Remark III.2.2.29 and the composite is an equivalence by the etale rigidity theorem proven by Hesselholt–Pstrągowski [HP25, Theorem 4.33]. So by 2-out-of-3 for equivalences, we see that
$$\pi_0^P \colon \operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})^{\mathrm{et}}_{\tau_{\geqslant \bullet} R/} \xrightarrow{\sim} \operatorname{DRing}^{\mathrm{et}}_{\pi_* R/}$$
is an equivalence. $\square$

Remark III.2.2.31. A much more general etale rigidity theorem was proven by Jiachen Liang in [Lia26, Theorem 6.26], which also applies to this setting. Since the proof was very simple in the case of filtered commutative algebras in the image of $\tau_{\geqslant \bullet} \colon \operatorname{CAlg}(\mathcal{S}p) \to \operatorname{CAlg}(\mathcal{S}p^{\mathrm{fil}})$, we included it in this paper. The case of filtered commutative ring spectra in the image of the Postnikov tower is also the only case we need for the purposes of this paper

III.2.3 Digression: The spectrally flat topology on filtered ring spectra

There is a finer topology on $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})$ which also deserves to be called the *flat topology* on $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})$. In this section we will introduce this topology, and in the name of being thorough we will restate some of the results from Section III.2.2, that have analogous statements for this topology. This is not meant to be a complete exposition of this subject.

Definition III.2.3.1. A map of filtered ring spectra $f\colon R \to S$ is *spectrally flat* if $\pi_{0,0}R \to \pi_{0,0}S$ is a flat map of commutative rings and for every $n, k \in \mathbb{Z}$ the canonical map

$$\pi_{n,k}R \underset{\pi_{0,0}R}{\otimes} \pi_{0,0}S \to \pi_{n,k}S$$

is an equivalence. We say that $f\colon R \to S$ is *spectrally faithfully flat* if it is spectrally flat and the functor $S \otimes_R -$ is conservative.

A finite collection $\{R \to S_i\}_{i\in I}$ is a *spectrally flat cover* if the map

$$R \to \prod_{i\in I} S_i$$

is spectrally faithfully flat.

Remark III.2.3.2. If a map of filtered ring spectra is spectrally (faithfully) flat, then it is faithfully flat. In particular, every spectrally flat cover is a flat cover.

Proposition III.2.3.3. *Let R be a connective filtered ring spectrum, in this situation there exists a finitary Grothendieck topology on $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})^{\mathrm{op}}_R$, such that, if $A \in \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})_R$, then a sieve $\mathcal{C} \subseteq (\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})^{\mathrm{op}}_R)_{/A}$ is a covering sieve if and only if it contains a spectrally flat cover of A.*

Proof. The proof is essentially the same as in Proposition III.2.2.15, one just has to notice that the t-structure homotopy groups of a filtered spectrum are intimately related to bigraded homotopy groups, and carefully take care of the extra bookkeeping. $\square$

Proposition III.2.3.4. *Let R be a connective filtered ring spectrum and $\mathcal{C}$ is a category with all limits. A functor $F\colon \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})_R \to \mathcal{C}$ is a sheaf for the spectrally flat topology if and only if the following conditions are satisfied.*

1. *F preserves finite products.*

2. *For every spectrally flat cover $R \to S$ the canonical map*

$$F(R) \to \lim_{[n]\in\Delta} F(S^{\otimes_R n+1})$$

   is an equivalence.

Proof. This is exactly as in Proposition III.2.2.17. $\square$

Proposition III.2.3.5. *A map of ring spectra $f\colon R \to S$ is (faithfully) flat if and only if the map $\tau_{\geqslant\bullet}f\colon \tau_{\geqslant\bullet}R \to \tau_{\geqslant\bullet}S$ is spectrally (faithfully) flat.*

Proof. The argument is essentially the same as in Proposition III.2.2.21. $\square$

Corollary III.2.3.6. *The functor $\tau_{\geqslant\bullet}\colon \mathrm{CAlg} \to \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})$ takes flat covers to spectrally flat covers.*

III.2.4 Faithfully flat descent

In this section we will prove faithfully flat descent for connective filtered ring spectra. This follows the proof of faithfully flat descent for ring spectra by Lurie in [Lurb, Appendix D], and is our stepping stone into a well functioning theory of filtered stacks. We want to add that this section can be simplified significantly by using new technology introduced by Burklund in forthcoming work on descent in general topologies. We will touch briefly on this simplification in Section III.2.5.

Remark III.2.4.1. It follows from [Lur17b, Theorem 4.5.3.1] that for any presentably symmetric monoidal ∞-category $\mathcal{C}$ there is a functor $\mathrm{Mod}\colon \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathrm{Pr}^L)$ which on objects is given by $R \mapsto \mathrm{Mod}_R(\mathcal{C})$ and by base-change on morphisms. The goal of this section will be to see that this functor is a sheaf for the faithfully flat topology on $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})$.

Notation III.2.4.2. In this situation of Remark III.2.4.1, for any map of algebras $f\colon A \to B \in \mathrm{CAlg}(\mathcal{C})$, we denote the induced functor $\mathrm{Mod}_A(\mathcal{C}) \to \mathrm{Mod}_B(\mathcal{C})$ by f^* and we denote its right adjoint, the forgetful functor, by f_*.

Theorem III.2.4.3. The functor $\mathrm{Mod}\colon \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0}) \to \mathrm{CAlg}(\mathrm{Pr}^L)$ is a sheaf for the faithfully flat topology on $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})^{\mathrm{op}}$. More precisely, the following two conditions are satisfied:

1. The functor $\mathrm{Mod}\colon \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0}) \to \mathrm{CAlg}(\mathrm{Pr}^L)$ preserves finite products.

2. For every faithfully flat cover $R \to S$ of filtered ring spectra, the canonical map
$$\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}}) \to \lim_{[n]\in\Delta} \mathrm{Mod}_{S^{\otimes_R n+1}}(\mathcal{S}p^{\mathrm{fil}})$$
is an equivalence.

Lemma III.2.4.4. *The functor* $\mathrm{Mod}\colon \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0}) \to \mathrm{CAlg}(\mathrm{Pr}^L)$ *preserves finite products.*

Proof. We may reduce to the case where $A \simeq A_1 \times A_2 \in \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})$. We want to show that the functor

$$p\colon \mathrm{Mod}_A(\mathcal{S}p^{\mathrm{fil}}) \to \mathrm{Mod}_{A_1}(\mathcal{S}p^{\mathrm{fil}}) \times \mathrm{Mod}_{A_2}(\mathcal{S}p^{\mathrm{fil}})$$

induced by base-change along $p_i\colon A \to A_i$, is an equivalence. Since p_i^* both admit right adjoints, it follows from [HY17, Theorem 5.5] that p admits a right adjoint given by

$$p^R = p_{1*} \times p_{2*},$$

where p_{i*} is the forgetful functor $\mathrm{Mod}_{A_i}(\mathcal{S}p^{\mathrm{fil}}) \to \mathrm{Mod}_A(\mathcal{S}p^{\mathrm{fil}})$. It suffices to prove the following two statements.

1. The functor p is conservative.
2. The functor p^R is fully faithful.

We first show (1). This is clear since $\mathrm{id} \simeq p^R p$, and hence p is fully faithful and thus conservative.

For (2), let $(M_1, M_2) \in \mathrm{Mod}_{A_1}(\mathcal{S}p^{\mathrm{fil}}) \times \mathrm{Mod}_{A_2}(\mathcal{S}p^{\mathrm{fil}})$. We will show that the counit $pp^R(M_1, M_2) \to M$ is an equivalence. Note, that it suffices to show that the map

$$p_i^* p^R(M_1, M_2) \to M_i$$

is an equivalence for $i \in \{1, 2\}$, that is the map $A_i \otimes_A (M_1 \times M_2) \to M_i$ is an equivalence. Without loss of generality we prove this for $i = 1$. To see this, we start by examining the left-hand side:

$$\begin{aligned} A_1 \underset{A}{\otimes} (M_1 \times M_2) &\simeq (A_1 \underset{A}{\otimes} A_1 \underset{A_1}{\otimes} M_1) \times (A_1 \underset{A}{\otimes} A_2 \underset{A_2}{\otimes} M_2) \\ &\simeq (A_1 \underset{A}{\otimes} A_1 \underset{A_1}{\otimes} M_1) \\ &\simeq A_1 \underset{A_1}{\otimes} M_1 \\ &\simeq M_1. \end{aligned}$$

Here the first equivalence follows from the equivalence $A \simeq A_1 \times A_2$. The second equivalence follows from the fact that $A_1 \otimes_A A_2 \simeq 0$. This can be seen using the Künneth spectral sequence to deduce that $\pi_0^P(A_1 \otimes_A A_2) \cong \pi_0^P A_1 \otimes_{\pi_0^P A} \pi_0^P A_2$. Now, since π_0^P preserves finite products, we conclude that $\pi_0^P(A_1 \otimes_A A_2) \simeq 0$. As a consequence $\pi_*^P(A_1 \otimes_A A_2)$ must be trivial. Finally, the third equivalence follows from the fact that A_1 is an idempotent A-algebra. This can be seen from the following computation:

$$\begin{aligned} A_1 \underset{A}{\otimes} A_1 &\simeq A_1 \underset{A}{\otimes} A_1 \times A_1 \underset{A}{\otimes} A_2 \\ &\simeq A_1 \underset{A}{\otimes} (A_1 \times A_2) \\ &\simeq A_1. \end{aligned}$$

This completes the proof. □

Lemma III.2.4.5. *Let $f\colon A \to B$ be a map of connective filtered ring spectra. The following conditions are equivalent:*

1. *The functor*

$$f^*\colon \operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}}) \to \operatorname{Mod}_B(\mathcal{S}p^{\mathrm{fil}})$$

is comonadic.

2. *The functor*

$$\operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}}) \to \lim_{[n]\in\Delta} \operatorname{Mod}_{B^{\otimes_A n+1}}(\mathcal{S}p^{\mathrm{fil}})$$

is an equivalence.

Proof. This is a standard argument and is completely analogous to [Lurb, Lemma D.3.5.7]. □

Remark III.2.4.6. In order to prove Theorem III.2.4.3 we are now reduced to showing that for every faithfully flat map $f\colon A \to B$ of connective filtered ring spectra, the functor

$$f^*\colon \operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}}) \to \operatorname{Mod}_B(\mathcal{S}p^{\mathrm{fil}})$$

is comonadic. By the Barr–Beck–Lurie theorem [Lur17b, Theorem 4.7.3.5] this is equivalent to showing that f^* is conservative and that for every f^*-split cosimplicial object $M_\bullet\colon \Delta \to \operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}})$, the limit exists and is preserved by f^*. Note that since $\operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}})$ is presentable, the limit exists, so we are reduced to seeing that the assembly map

$$\phi_M\colon f^*(\operatorname{Tot}(M_\bullet)) \to \operatorname{Tot}(f^*(M_\bullet))$$

is an equivalence. We first prove this in the special case where $M_\bullet$ takes values in $\operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}})^\heartsuit$.

Lemma III.2.4.7. *Let $f\colon A \to B$ be a faithfully flat map of connective filtered ring spectra and let $M_\bullet\colon \Delta \to \operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}})$ be a f^*-split cosimplicial object. In this situation, if $M_\bullet$ takes values in $\operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}})^\heartsuit$, then the totalization $\operatorname{Tot}(f^*(M_\bullet))$ is discrete and the augmentation map*

$$\phi_M\colon f^*(\operatorname{Tot}(M_\bullet)) \to \operatorname{Tot}(f^*(M_\bullet))$$

is an equivalence.

Proof. Let $f\colon A \to B$ be a faithfully flat map of connective filtered ring spectra and let $M_\bullet$ be a discrete f^*-split cosimplicial object. Since $M_\bullet$ is f^*-split, the cosimplicial object $\pi_k^P f^*(M_\bullet)$ is split for every k and hence

$$\pi_k^P \operatorname{Tot}(f^*(M_\bullet)) \to \operatorname{Tot}(\pi_k^P f^*(M_\bullet))$$

is an equivalence. In particular, since f^* is t-exact we have that

$$\begin{aligned}\pi_k^P \operatorname{Tot}(f^*(M_\bullet)) &\cong \operatorname{Tot}(\pi_k^P f^*(M_\bullet))\\ &\cong \operatorname{Tot}(f^*(\pi_k^P M_\bullet))\\ &\cong \begin{cases}\operatorname{Tot}(f^*(\pi_0^P M_\bullet)) & k=0\\ 0 & \text{else.}\end{cases}\end{aligned}$$

It remains to be checked that $\phi_M \colon f^*(\lim_{[k]\in\Delta}(M_k)) \to \lim_{[k]\in\Delta}(f^*(M_k))$ is an equivalence. This follows immediately from Lemma A.1.33 since B is assumed to be flat, and thus has Tor-amplitude 0. □

Proof of Theorem III.2.4.3. Let $f\colon A \to B$ be a faithfully flat cover of connective filtered ring spectra. By combining Lemma III.2.4.4, Lemma III.2.4.5 and the Barr–Beck–Lurie theorem we are reduced to showing the following two statements:

1. The functor $f^*\colon \operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}}) \to \operatorname{Mod}_B(\mathcal{S}p^{\mathrm{fil}})$ is conservative.

2. If $M^\bullet\colon \Delta \to \operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}})$ is a f^*-split cosimplicial diagram, then the canonical augmentation map

$$f^*(\operatorname{Tot} M^\bullet) \to \operatorname{Tot} f^*(M^\bullet)$$

   is an equivalence.

The claim (1) follows from our assumption that f is faithful, so only (2) remains to be proven.

Let $M\colon \Delta \to \operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}})$ be any f^*-split cosimplicial object. We claim that we can reduce to the case where M is discrete. For all $a \leqslant b \in \mathbb{Z}$ we define $M_{[a,b]} := \tau_{\leqslant b}\tau_{\geqslant a} M$ to be the postcomposition of M with $\tau_{\leqslant b}\tau_{\geqslant a}\colon \operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}}) \to \operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}})$. Since f^* is t-exact, we see that $f^*(M_{[a,b]}) \simeq f^*(M)_{[a,b]}$. Since the diagonal t-structure on $\mathrm{Mod}_A(\mathcal{S}p^{\mathrm{fil}})$ (and Mod_B) is left and right complete, we obtain an equivalence

$$\mathrm{colim}_{a\to-\infty}\lim_{b\to\infty} M_{[a,b]} \simeq \mathrm{colim}_{a\to-\infty}\,\tau_{\geqslant a}M \simeq M.$$

Similarly, we obtain equivalences

$$\begin{aligned}
f^*(M) &\simeq f^*(\mathrm{colim}_{a\to-\infty}\lim_{b\to\infty} M_{[a,b]})\\
&\simeq \mathrm{colim}_{a\to-\infty}\, f^*(\lim_{b\to\infty} M_{[a,b]})\\
&\simeq \mathrm{colim}_{a\to-\infty}\lim_{b\to\infty} f^*(M_{[a,b]})\\
&\simeq \mathrm{colim}_{a\to-\infty}\lim_{b\to\infty} f^*(M)_{[a,b]}\\
&\simeq f^*(M).
\end{aligned}$$

Here the first equivalence follows from our computation above, the second follows from f^* preserving colimits, the third follows from the fact that f^* is left t-exact; as a consequence the fiber

$$\mathrm{fib}(f^*(M_{[a,b+1]}) \to f^*(M_{[a,b]}))$$

is discrete and concentrated in degree $b+1$. Hence, the limit

$$\lim_{b\to\infty}\mathrm{fib}(f^*(M_{[a,b+1]}) \to f^*(M_{[a,b]})) \simeq 0$$

vanishes. The fourth equivalence we proved above and the fifth equivalence is the same argument as the first.

In order to finish the proof we proceed via induction on $n = b-a$. The case $n=0$ is proven in Lemma III.2.4.7, so it suffices to prove the induction step. To see this we consider the fiber sequence

$$M_{[b+1,b+1]} \to M_{[a,b+1]} \to M_{[a,b]},$$

which induces a map of fiber sequences

$$\begin{tikzcd}
f^*(\lim_{[k]\in\Delta} M^k_{[b+1,b+1]}) \arrow[r] \arrow[d, "\simeq"] & f^*(\lim_{[k]\in\Delta} M^k_{[a,b+1]}) \arrow[r] \arrow[d] & f^*(\lim_{[k]\in\Delta} M^k_{[a,b]}) \arrow[d, "\simeq"] \\
\lim_{[k]\in\Delta} f^*(M^k_{[b+1,b+1]}) \arrow[r] & \lim_{[k]\in\Delta} f^*(M^k_{[a,b+1]}) \arrow[r] & \lim_{[k]\in\Delta} f^*(M^k_{[a,b]}).
\end{tikzcd}$$

Here the left and right vertical maps are equivalences by our induction hypothesis, and we conclude that the same holds for the middle vertical map. □

Corollary III.2.4.8. *For any connective filtered ring spectrum R, let $\mathrm{Mod}^\flat_R(\mathcal{S}p^{\mathrm{fil}})$ denote the full subcategory of $\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$ spanned by modules which are flat over R. If $R \to S$ is faithfully flat, then the equivalence*

$$\phi\colon\ \mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}}) \to \lim_{[k]\in\Delta} \mathrm{Mod}_{S^{\otimes_R k+1}}(\mathcal{S}p^{\mathrm{fil}})$$

restricts to an equivalence

$$\mathrm{Mod}^\flat_R(\mathcal{S}p^{\mathrm{fil}}) \to \lim_{[k]\in\Delta} \mathrm{Mod}^\flat_{S^{\otimes_R k+1}}(\mathcal{S}p^{\mathrm{fil}}).$$

In particular, an R-algebra R' is (faithfully) flat if and only if the algebra $R' \otimes_R S^{\otimes_R k+1}$ is (faithfully) flat over $S^{\otimes_R k+1}$ for every k.

Proof. It is clear that every map of filtered ring spectra carries flat modules to flat modules. In particular, ϕ restricts to a fully faithful functor

$$\phi^\flat\colon\ \mathrm{Mod}^\flat_R(\mathcal{S}p^{\mathrm{fil}}) \hookrightarrow \lim_{[k]\in\Delta} \mathrm{Mod}^\flat_{S^{\otimes_R k+1}}(\mathcal{S}p^{\mathrm{fil}}).$$

We claim that $\phi^\flat$ is also essentially surjective. It suffices to see that a filtered R-module M is (faithfully) flat if and only if $S \otimes_R M$ is faithfully flat. It follows from the above that if M is (faithfully) flat, then so is $S \otimes_R M$. Suppose that $S \otimes_R M$ is (faithfully) flat. Since $R \to S$ is faithfully flat, we know that $M \otimes_R N \simeq 0$ if and only if $S \otimes_R M \otimes_R N \simeq S \otimes_R M \otimes_S S \otimes_R N \simeq 0$. It remains to be shown that M is flat. By assumption, $\pi_0^P(S \otimes_R M)$ is flat over $\pi_0^P S$, so $\pi_0^P M$ is flat over $\pi_0^P R$ by [HP25, Lemma 3.18]. We will now show that the map

$$\pi_*^P R \underset{\pi_0^P R}{\otimes} \pi_0^P M \to \pi_*^P M$$

is an isomorphism. It follows from our assumptions that $\pi_*^P R \to \pi_*^P S$ is a faithful, so it suffices to see that

$$\pi_*^P S \underset{\pi_0^P R}{\otimes} \pi_0^P M \simeq \pi_*^P S \underset{\pi_*^P R}{\otimes} \pi_*^P R \underset{\pi_0^P R}{\otimes} \pi_0^P M \to \pi_*^P S \underset{\pi_*^P R}{\otimes} \pi_*^P M$$

is an isomorphism. To see this consider the diagram

$$\begin{tikzcd}
\pi_*^P S \otimes_{\pi_*^P R} \pi_*^P R \otimes_{\pi_0^P R} \pi_0^P M \arrow[r, "\sim"] \arrow[d] & \pi_*^P S \otimes_{\pi_0^P R} \pi_0^P M \arrow[d] & \pi_*^P R \otimes_{\pi_0^P R} \pi_0^P S \otimes_{\pi_0^P R} \pi_0^P M \arrow[l, "\sim"'] \arrow[d, "\sim"] \\
\pi_*^P S \otimes_{\pi_*^P R} \pi_*^P M \arrow[r, "\sim"] & \pi_*^P (S \otimes_R M) & \pi_*^P R \otimes_{\pi_0^P R} \pi_0^P (S \otimes_R M) \arrow[l, "\sim"']
\end{tikzcd}$$

Here the top horizontal maps are equivalences, definitionally since S is flat over R. The left horizontal map in the bottom is an equivalence by Proposition III.2.2.9 since S is flat over R. Furthermore, the right horizontal map in the bottom is an equivalence since $S \otimes_R M$ is flat over S and thus by Lemma III.2.2.4 flat over R, since S is flat over R. Finally, the right vertical map is an equivalence by since R is connective, and hence so are S and M. The case of M can be verified as follows. By assumption $S \otimes_R M$ is flat over S and hence connective. In particular M must be connective, since $S \otimes_R -$ is t-exact and detects connectivity. In particular, if one inspects the Künneth spectral sequence for $S \otimes_R M$, one sees that $\pi_0^P(S \otimes_R M) \cong \pi_0^P S \otimes_{\pi_0^P R} \pi_0^P M$. We deduce, by 2-out-of-3 that the remaining vertical maps are equivalences. $\square$

III.2.5 Digression: π_*-flat descent of ring spectra

The results of Section III.2.4 allow us to generalize Lurie's theorem of faithfully flat descent for ring spectra [Lurb, Theorem D.6.3.1][1], to the case of maps $R \to S$ which are faithful and π_*-flat. The results of this section have been proven separately by Robert Burklund using even stronger descent technology.

Proposition III.2.5.1 ([Bar23, Lemma 2.23])**.** *The functors symmetric monoidal functors $\mathcal{S}p^{\mathrm{fil}} \xrightarrow{\tau^{-1}} \mathcal{S}p$ and $\mathcal{S}p^{\mathrm{fil}} \xrightarrow{\mathrm{gr}_*} \mathcal{S}p^{\mathrm{gr}}$ provide $\mathcal{S}p$ and $\mathcal{S}p^{\mathrm{gr}}$ with canonical $\mathcal{S}p^{\mathrm{fil}}$-module structures in Pr^L, such that $\mathcal{S}p$ and $\mathcal{S}p^{\mathrm{gr}}$ are both dualizable in $\mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L)$. In particular, the functors*

$$\mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L) \to \mathrm{Pr}^L \qquad \mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L) \to \mathrm{Mod}_{\mathcal{S}p^{\mathrm{gr}}}(\mathrm{Pr}^L),$$

sending a $\mathcal{S}p^{\mathrm{fil}}$-module M to $\mathcal{S}p \otimes_{\mathcal{S}p^{\mathrm{fil}}} M$ and $\mathcal{S}p^{\mathrm{gr}} \otimes_{\mathcal{S}p^{\mathrm{fil}}} M$ respectively, preserves limits.

Remark III.2.5.2. Since $\tau^{-1} \colon \mathcal{S}p^{\mathrm{fil}} \to \mathcal{S}p$ is a smashing localization we see that $\mathcal{S}p \simeq \mathrm{Mod}_{\tau^{-1}\mathbb{S}^{0,0}}(\mathcal{S}p^{\mathrm{fil}})$. In particular, if $R \in \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})$ is a commutative filtered ring spectrum, then we have equivalences

$$\begin{aligned} \mathcal{S}p \underset{\mathcal{S}p^{\mathrm{fil}}}{\otimes} \mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}}) &\simeq \mathrm{Mod}_{\tau^{-1}\mathbb{S}^{0,0}}(\mathcal{S}p^{\mathrm{fil}}) \underset{\mathcal{S}p^{\mathrm{fil}}}{\otimes} \mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}}) \\ &\simeq \mathrm{Mod}_{\tau^{-1}\mathbb{S}^{0,0} \otimes R}(\mathcal{S}p^{\mathrm{fil}}) \\ &\simeq \mathrm{Mod}_{\tau^{-1}R}(\mathcal{S}p). \end{aligned}$$

In particular, if $R = \tau_{\geqslant \bullet} S$ for some ring spectrum S, then we have an equivalence $\mathcal{S}p \otimes_{\mathcal{S}p^{\mathrm{fil}}} \mathrm{Mod}_{\tau_{\geqslant \bullet} S}(\mathcal{S}p^{\mathrm{fil}}) \simeq \mathrm{Mod}_S(\mathcal{S}p)$ in Pr^L. This is explained in greater detail in [BHS26, Appendix A, Appendix B].

[1] The cited theorem actually proves faithfully flat hyperdescent, we are only generalizing the descent part of the statement

Theorem III.2.5.3. If $R \to S$ is a faithful and π_*-flat map, then the functor
$$\mathrm{Mod}_R(\mathcal{S}p) \to \lim_{k \in \Delta} \mathrm{Mod}_{S^{\otimes_R k+1}}(\mathcal{S}p)$$
is an equivalence.

Proof. It follows from Proposition III.2.2.21 that the induced map $\tau_{\geqslant \bullet} R \to \tau_{\geqslant \bullet} S$ is a faithfully flat map of filtered ring spectra. Therefore, by Theorem III.2.4.3 we get that the functor
$$\mathrm{Mod}_{\tau_{\geqslant \bullet} R}(\mathcal{S}p^{\mathrm{fil}}) \to \lim_{k \in \Delta} \mathrm{Mod}_{\tau_{\geqslant \bullet} S^{\otimes_{\tau_{\geqslant \bullet} R} k+1}}(\mathcal{S}p^{\mathrm{fil}})$$
is an equivalence. Hence, following the computations of Remark III.2.5.2 we see that the induced map
$$\mathrm{Mod}_R(\mathcal{S}p) \xrightarrow{\sim} \lim_{k \in \Delta} \mathrm{Mod}_{S^{\otimes_R k+1}}(\mathcal{S}p)$$
is an equivalence. It can be checked, using that $\tau_{\geqslant \bullet}$ is a section of τ^{-1}, that the equivalence above agrees with the functor induced by base-change along $R \to S$. □

Remark III.2.5.4. In [Mat16, Section 3.3] Mathew introduces the notion of the descendable maps of commutative algebra objects in a presentably symmetric monoidal stable category $\mathcal{C}$ and proves that the Grothendieck topology on $\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$ associated to this class of maps is the coarsest topology for which the functor
$$\mathrm{CAlg}(\mathcal{C}) \to \widehat{\mathrm{Cat}}$$
sending $R \in \mathrm{CAlg}(\mathcal{C})$ to $\mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^L)$ is a sheaf. Unfortunately, not all flat maps of (filtered) ring spectra are descendable cf. [Aok24, Theorem 3.15], so this does not provide a proof of faithfully flat descent. However, there is a coarser topology, introduced by Burklund, called

the *phantom descendable* topology and it is the coarsest topology on $\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$ for which the functor

$$\mathrm{CAlg}(\mathcal{C}) \xrightarrow{\mathrm{Mod}_{(-)}(\mathcal{C})} \widehat{\mathrm{Cat}}$$

is a sheaf. In fact, one can show using a version of Lazard's theorem that every faithfully flat map of (filtered) ring spectra is phantom descendable. We will not pursue this avenue in this paper, but hope to do so in the future.

Chapter III.3

Filtered stacks: stacks

III.3.1 Filtered stacks

We will now produce a general theory of *filtered stacks* building on the results of Chapter III.2. Since the category $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})$ is large we will use the language of stacks on large categories introduced by Balderrama–Davies–Linskens in [BDL25, Part 2]. There are multiple ways to solve the set-theoretical issues at play when considering stacks on large categories. We choose this one, since it is the most well-suited for our purposes.

The following definition is a special case of [BDL25, Definition 2.1.1.5] in the case of FAff.

Definition III.3.1.1. We define the category of *filtered stacks* to be the colimit

$$\mathrm{FStack} := \mathrm{colim}_{\mathcal{C}_0 \subseteq \mathrm{FAff}} \mathrm{Shv}(\mathcal{C}_0).$$

indexed over all small subsites of FAff. Here FAff denotes the site $(\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0}), \mathrm{flat})$ as defined in Definition III.2.2.16.

Remark III.3.1.2. Balderrama–Davies–Linskens assume the topology in question is subcanonical; in the case of FAff this is a consequence of faithfully flat descent Theorem III.2.4.3.

The above definition of stacks is compatible with the usual universal property of sheaves.

Proposition III.3.1.3 ([BDL25, Eq. 2.1.1.8, Prop. 2.1.2.2])**.** *There is a well-defined Yoneda embedding* $\mathrm{Spec}\colon \mathrm{FAff} \to \mathrm{FStack}$*. Furthermore, if* $\mathcal{D}$ *is a category with small limits, then restriction along* Spec *defines a fully faithful functor*

$$\mathrm{Fun}^R(\mathrm{FStack}, \mathcal{D}) \to \mathrm{Fun}(\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0}), \mathcal{D}),$$

with essential image given by the sheaves for the flat topology.

Definition III.3.1.4. A filtered stack X is *affine* if it is in the essential image of $\mathrm{Spec}\colon \mathrm{FAff} \to \mathrm{FStack}$.

Proposition III.3.1.5. *The spectrum functor* $\mathrm{Spec}\colon \mathrm{FAff} \to \mathrm{FStack}$ *is fully faithful.*

Proof. Since the flat topology on FAff is subcanonical, the Yoneda embedding defines a fully faithful functor $\mathcal{C}_0 \to \mathrm{Shv}(\mathcal{C}_0)$ for every small subsite $\mathcal{C}_0 \subseteq \mathrm{FAff}$. Since fully faithful functors are closed under filtered colimits in $\mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}})$, we conclude that $\mathrm{Spec}\colon \mathrm{FAff} \to \mathrm{FStack}$ is fully faithful. □

Remark III.3.1.6. It follows from Proposition III.2.1.23 that the category $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})$ is accessible, and we deduce from [BDL25, Cor. 2.1.2.5] that there is an equivalence

$$\mathrm{colim}_\kappa \mathrm{Shv}(\mathrm{FAff}_\kappa) \xrightarrow{\sim} \mathrm{FStack}\,.$$

Here the colimit is indexed over all sufficiently large regular cardinals κ and FAff_κ denotes the full subcategory of κ-cocompact objects.

Remark III.3.1.7. It follows from Proposition III.2.1.23 that the FAff is coaccessible. In particular, the argument from Remark A.2.12 applies, so every filtered stack can be written as a colimit of affines.

Proposition III.3.1.8. *The functor* $\tau_{\geqslant\bullet}\colon \mathrm{CAlg} \to \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})$ *is a map of sites.*

Proof. It follows from Corollary III.2.2.22 that $\tau_{\geqslant\bullet}$ preserves flat covers, so it suffices to prove that $\tau_{\geqslant\bullet}$ preserves pushouts along flat maps, which we prove in Lemma III.2.2.20. □

Theorem III.3.1.9. There is a colimit preserving functor

$$(-)^{\tau}\colon \mathrm{Stk} \to \mathrm{FStack},$$

such that $(\mathrm{Spec}\, R)^{\tau} \simeq \mathrm{Spec}(\tau_{\geqslant\bullet}R)$.

Proof. The existence part follows from [BDL25, Prop. 2.1.2.6] applied to the map of sites $\tau_{\geqslant\bullet}\colon \mathrm{Aff} \to \mathrm{FAff}$. In the proof of [BDL25, Prop. 2.1.2.6] the functor is constructed as the colimit of a natural transformations of functors all of which are compatible with the Yoneda embedding, it follows that the functor is compatible with Spec. □

Theorem III.3.1.10. The functor $\pi_0^P\colon \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0}) \to \mathrm{DRing}$ induces an exact map of sites

$$\mathrm{FAff} \to \mathrm{DAff}\,.$$

In particular, there exists a left exact colimit preserving functor

$$(-)^{\heartsuit}\colon \mathrm{FStack} \to \mathrm{DStk}\,.$$

Furthermore, for any affine filtered stack $\mathrm{Spec}\, R$, we have a canonical equivalence $(\mathrm{Spec}\, R)^{\heartsuit} \simeq \mathrm{Spec}\, \pi_0^P R$.

Proof. Let $R \to S$ be a flat map of connective filtered ring spectra, in this situation $\pi_0^P R \to \pi_0^P S$ is flat by definition, and it follows by inspecting the Künneth spectral sequence that $\pi_0^P(S \otimes_R M) \simeq \pi_0^P S \otimes_{\pi_0^P R} \pi_0^P M$ for any connective filtered R-module M. Thus, the functor $(-)^\heartsuit \colon \mathrm{FAff} \to \mathrm{DAff}$ preserves covers and pullbacks. So it suffices to prove that it preserves the terminal object. This is clear since $\pi_0^P \mathbb{S}^{0,0} = \mathbb{Z}[0]$. Finally, the identification $(\operatorname{Spec} R)^\heartsuit \simeq \operatorname{Spec} \pi_0^P R$ is done similarly to the proof of Theorem III.3.1.9. $\square$

Notation III.3.1.11. If $X = \operatorname{Spec} R$ is an affine spectral stack, we use will usually use the notation $\operatorname{Spec}^\tau R \coloneqq (\operatorname{Spec} R)^\tau = \operatorname{Spec} \tau_{\geqslant \bullet} R$. Similarly, we use the notation $\operatorname{Spec}^\heartsuit R \coloneqq (\operatorname{Spec} R)^\heartsuit$ for any connective filtered ring spectrum R. If R is any commutative ring spectrum, then we abuse notation and write $\operatorname{Spec}^\heartsuit R \coloneqq (\operatorname{Spec}^\tau R)^\heartsuit$, for the associated affine Dirac stack.

Definition III.3.1.12. A map of filtered stacks $f \colon X \to Y$ is *affine* if for every affine $\operatorname{Spec} R$ and map $\operatorname{Spec} R \to Y$ the pullback $\operatorname{Spec} R \times_Y X$ is an affine filtered stack. Furthermore, a map $f \colon X \to Y$ is *flat* if it is affine and for every affine $\operatorname{Spec} R$ and map $\operatorname{Spec} R \to Y$ the induced map $\operatorname{Spec} S \simeq \operatorname{Spec} R \times_Y X \to \operatorname{Spec} R$ induces a flat map of filtered ring spectra $R \to S$. A map of filtered stacks $X \to Y$ is *faithfully flat*[1] if it is flat and an effective epimorphism.

Remark III.3.1.13. It is clear from their definition that flat and affine maps are closed under pullbacks. We now prove the following useful fact giving a condition under which a map, which pulls back to an affine map, is affine. The result is completely analogous to [Lurb, Prop. 9.3.1.1].

[1]We will also sometimes say such a map is a (faithfully) flat cover.

Lemma III.3.1.14. *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a pullback square of filtered stacks. If f' is affine and g is an effective epimorphism, then f is affine. Furthermore, if f' is flat, then so is f.

Proof. Let $\operatorname{Spec} R \to Y$ be any map. We need to show that $\operatorname{Spec} R \times_Y X$ is affine. Since $g\colon Y' \to Y$ is an effective epimorphism, it follows from [HP24, Cor. B.7] that there exists a faithfully flat map of filtered ring spectra $R \to R_0$ and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R_0 \times_{Y'} X' \arrow[rr] \arrow[dr] \arrow[dd] & & X' \arrow[dr, "f'"] \arrow[dd] & \\ & \operatorname{Spec} R_0 \arrow[rr, "g'"] \arrow[dd] & & Y' \arrow[dd, "g"] \\ \operatorname{Spec} R \times_Y X \arrow[rr] \arrow[dr] & & X \arrow[dr, "f"] & \\ & \operatorname{Spec} R \arrow[rr] & & Y. \end{tikzcd}$$

The map $h\colon \operatorname{Spec} R_0 \to \operatorname{Spec} R$ is an effective epimorphism by faithfully flat descent. Now, let $R^\bullet$ denote the Čech nerve of $R \to R_0$ and consider $\operatorname{Spec} R^m \times_Y X$. Using the pasting laws for pullbacks we see that

$$\operatorname{Spec} R^m \times_Y X \simeq \operatorname{Spec} R^{m+1} \times_{Y'} X'.$$

By assumption $f'\colon X' \to Y'$ is affine, so $\operatorname{Spec} R^m \times_Y X \simeq \operatorname{Spec} A^m$ for some R^m-algebra A^m. Using faithfully flat descent of $R \to R_0$, we see that $A^m \simeq A \otimes_R R^m$ for some R-algebra A[1]. Since colimits are universal in FStack, we conclude that

$$\begin{aligned}
\operatorname{Spec} R \times_Y X &\simeq (\operatorname{colim}_{[k]\in\Delta^{\mathrm{op}}} \operatorname{Spec} R^k) \times_Y X \\
&\simeq \operatorname{colim}_{[k]\in\Delta^{\mathrm{op}}} (\operatorname{Spec} R^k \times_Y X) \\
&\simeq \operatorname{colim}_{[k]\in\Delta^{\mathrm{op}}} \operatorname{Spec} A^k \\
&\simeq \operatorname{colim}_{[k]\in\Delta^{\mathrm{op}}} \operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} R^k \\
&\simeq \operatorname{Spec} A \times_{\operatorname{Spec} R} (\operatorname{colim}_{[k]\in\Delta^{\mathrm{op}}} \operatorname{Spec} R^k) \\
&\simeq \operatorname{Spec} A.
\end{aligned}$$

Here we have used in the first and last equivalence that $h\colon \operatorname{Spec} R_0 \to \operatorname{Spec} R$ is an effective epimorphism, and thus $\operatorname{colim}_{[k]\in\Delta^{\mathrm{op}}} \operatorname{Spec} R^k \simeq \operatorname{Spec} R$.

Finally, if f' is flat, then A^m is flat over R^m for all m, thus A is flat over R, which is what we wanted to prove. $\square$

Remark III.3.1.15. The proof of Lemma III.3.1.14 is completely analogous to the proof of [Lurb, Lemma 9.3.1.1]. We only spell it out here in order to clarify details that we struggled to understand when first learning of this result.

Corollary III.3.1.16. *Let X be a filtered stack and let $\mathrm{FStack}^{\mathrm{aff}}_{/X}$ denote the full subcategory of $\mathrm{FStack}_{/X}$ spanned by affine morphisms $Z \to X$. If $Y \to X$ is an effective epimorphism, then the equivalence*

$$\mathrm{FStack}_{/X} \xrightarrow{\sim} \lim_{[k]\in\Delta} \mathrm{FStack}_{/Y^{\times_X k+1}}$$

[1] To see this we emphasize that the system $\{A^m\}_{[m]\in\Delta}$ determines and object in the limit $\lim_{[k]\in\Delta}(\mathrm{CAlg}(\mathrm{Mod}_{R^m}(\mathcal{S}p^{\mathrm{fil}})))$, which is $\mathrm{CAlg}(\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}}))$ by faithfully flat descent.

restricts to an equivalence

$$\mathrm{FStack}^{\mathrm{aff}}_{/X} \xrightarrow{\sim} \lim_{[k]\in\Delta} \mathrm{FStack}^{\mathrm{aff}}_{/Y^{\times_X k+1}}.$$

Definition III.3.1.17. A filtered stack X is *geometric* if

1. there exist an affine filtered stack $\operatorname{Spec} R$ and a faithfully flat cover $\operatorname{Spec} R \to X$.

2. The diagonal $X \to X \times X$ is affine.

Remark III.3.1.18. The definition of geometric stack above differs from the definition in [BDL25, Definition 3.4.1.1], where they do not ask for condition (2) in Definition III.3.1.17. Our definition is in agreement with often what shows up more commonly in the literature. We choose to include this condition since it allows us to prove that the descent spectral sequence for a geometric stack does not depend on a choice of atlas, cf. [CDN25, Lemma 2.3][1]. Condition (2) also implies the following: Let $\operatorname{Spec} R \to X$ and $\operatorname{Spec} S \to X$ be any pair of maps into a geometric stack, then $\operatorname{Spec} R \times_X \operatorname{Spec} S$ is affine. To see this, consider the pullback diagram

$$\begin{tikzcd}
\operatorname{Spec} R \times_X \operatorname{Spec} S \arrow[r] \arrow[d] & \operatorname{Spec} R \times \operatorname{Spec} S \arrow[d] \\
X \arrow[r, "\Delta"] & X \times X.
\end{tikzcd}$$

The upper left corner is affine, since $\operatorname{Spec} R \times \operatorname{Spec} S \simeq \operatorname{Spec}(R \otimes S)$ and $\Delta\colon X \to X \times X$ is assumed to be an affine map.

[1]The cited result requires strictly stronger assumptions than what is present in our context, which is why we are purposefully vague. We will provide a precise statement later in the paper.

Remark III.3.1.19. If X is a geometric filtered stack and $p\colon \operatorname{Spec} R \to X$ and $q\colon \operatorname{Spec} S \to X$ are flat covers, then there exists a flat cover refining them in the following sense. Consider the pullback diagram

$$\begin{tikzcd}
\operatorname{Spec} R \times_X \operatorname{Spec} S \arrow[r] \arrow[d] & \operatorname{Spec} S \arrow[d, "q"] \\
\operatorname{Spec} R \arrow[r, "p"] & X,
\end{tikzcd}$$

Since p is affine, $\operatorname{Spec} R \times_X \operatorname{Spec} S \simeq \operatorname{Spec} A$ and the maps $R \to A$ and $S \to A$ are both flat. Indeed, they are actually faithfully flat, by our assumption that p and q are flat covers. In particular, the composite $\operatorname{Spec} A \to X$ is a faithfully flat cover. It follows from the above, that if $R = S$ and $p = q$, then $\operatorname{Spec} R_1 \simeq \operatorname{Spec} R \times_X \operatorname{Spec} R \to X$ is a faithfully flat cover and $R \to R_1$ is faithfully flat. Moreover, if $\operatorname{Spec} R_\bullet$ is the Čech nerve of $\operatorname{Spec} R \to X$, then the same conclusions hold for every face map of the simplicial diagram. This property of geometric stacks turns out to characterize them in the following sense.

Proposition III.3.1.20. *A filtered stack X is geometric if and only if there exists a groupoid object $\mathbb{X}\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{FStack}$ with $\operatorname{colim}_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{X} \simeq X$, and which factors through* FAff*: that is, for all $[k] \in \mathbb{\Delta}$*

$$\mathbb{X}_k := \mathbb{X}([k]) = \operatorname{Spec} A_k$$

for some connective filtered ring spectrum A_k and all the face maps are faithfully flat.

Proof. It follows from [BDL25, Prop. 2.1.2.9] that groupoids in FStack are effective, in particular if X is geometric and $p\colon \operatorname{Spec} R \to X$ is a faithfully flat cover, then the groupoid obtained by the Čech nerve of p satisfies the needed conditions by our considerations in Remark III.3.1.19. Likewise, if $\mathbb{X}\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{FStack}$ is a groupoid taking values in FAff with faithfully flat face maps, then $\mathbb{X}_0 \to \operatorname{colim}_{[k]\in\mathbb{\Delta}^{\mathrm{op}}} \mathbb{X}_k \simeq$

X is an effective epimorphism. It remains to be seen that $\mathbb{X}_0 \to X$ is flat. This follows immediately from the pullback diagram

$$\begin{tikzcd} \mathbb{X}_1 \arrow[r, "d_0"] \arrow[d, "d_1"] & \mathbb{X}_0 \arrow[d] \\ \mathbb{X}_0 \arrow[r] & X. \end{tikzcd}$$

Here, d_0 is faithfully flat by assumption and $\mathbb{X}_0 \to X$ is an effective epimorphism, thus it follows from Lemma III.3.1.14 that $\mathbb{X}_0 \to X$ is flat too. □

Definition III.3.1.21 ([BDL25, Definition 3.4.2.1])**.** A map of geometric filtered stacks $f\colon X \to Y$ is *geometrically flat* if for every pair of flat covers $\operatorname{Spec} R \to Y$ and $\operatorname{Spec} S \to X$ the map

$$\operatorname{Spec} A \simeq \operatorname{Spec} S \times_Y \operatorname{Spec} R \to \operatorname{Spec} R$$

induces a flat map $R \to A$ of filtered ring spectra. We say that f is faithfully geometrically flat if it is also an effective epimorphism.

Remark III.3.1.22. Not all geometrically flat maps of geometric filtered stacks are flat. For instance, they can fail to be affine, see [BDL25, Example 3.4.2.2] for a discussion in the non-filtered setting.

Lemma III.3.1.23. *If* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is a flat map of affine spectral stacks, then for any* $\operatorname{Spec} T \to \operatorname{Spec} R$*, the augmentation map*

$$(\operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} T)^{\tau} \to \operatorname{Spec}^{\tau} S \times_{\operatorname{Spec}^{\tau} R} \operatorname{Spec}^{\tau} T$$

is an equivalence.

Proof. Since $(\operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} T)^{\tau}$ and $\operatorname{Spec}^{\tau} S \times_{\operatorname{Spec}^{\tau} R} \operatorname{Spec}^{\tau} T$ are both affine, it suffices to see that the induced map on global sections is an equivalence. That is, we have to see that $\tau_{\geqslant \bullet} S \otimes_{\tau_{\geqslant \bullet} R} \tau_{\geqslant \bullet} T \to \tau_{\geqslant \bullet}(S \otimes_R T)$ is an equivalence. This follows from Lemma III.2.2.20, since $R \to S$ is flat. □

Lemma III.3.1.24. *If $X \to \operatorname{Spec} R$ is a geometrically flat map of geometric stacks, then for any affine $\operatorname{Spec} S$ and map $\operatorname{Spec} S \to \operatorname{Spec} R$ the canonical map*

$$(\operatorname{Spec} S \times_{\operatorname{Spec} R} X)^\tau \to \operatorname{Spec}^\tau S \times_{\operatorname{Spec}^\tau R} X^\tau$$

is an equivalence.

Proof. Let $\operatorname{Spec} R' \to X$ be a faithfully flat cover, such that we have $\operatorname{colim}_{[k]\in\Delta^{\mathrm{op}}} \operatorname{Spec} R'^{\times_X k+1} \simeq X$. Furthermore, let $Y := \operatorname{Spec} S \times_{\operatorname{Spec} R} X$ and let $Y_{R'}^{[k]} := Y \times_X \operatorname{Spec} R'^{\times_X k+1}$. Since $\operatorname{Spec} R' \to X$ is a faithfully flat cover, so is $Y_{R'}^{[0]} \to Y$ and thus

$$\operatorname{colim}_{[k]\in\Delta^{\mathrm{op}}} Y_{R'}^{[k]} \to Y$$

is an equivalence. Note that $Y_{R'}^{[k]} \simeq \operatorname{Spec} S \times_{\operatorname{Spec} R} (\operatorname{Spec} R'^{\times_X k+1})$ for every $k \in \Delta$. In particular, Lemma III.3.1.23 implies that

$$(Y_{R'}^{[k]})^\tau \simeq \operatorname{Spec}^\tau S \times_{\operatorname{Spec}^\tau R} (\operatorname{Spec} R'^{\times_X k+1})^\tau.$$

Putting it all together we see that

$$\begin{aligned}
Y^\tau &\simeq (\operatorname{colim}_{[k]\in\Delta^{\mathrm{op}}} Y_{R'}^{[k]})^\tau \\
&\simeq \operatorname{colim}_{[k]\in\Delta^{\mathrm{op}}} (Y_{R'}^{[k]})^\tau \\
&\simeq \operatorname{colim}_{[k]\in\Delta^{\mathrm{op}}} \operatorname{Spec}^\tau S \times_{\operatorname{Spec}^\tau R} (\operatorname{Spec} R'^{\times_X k+1})^\tau \\
&\simeq \operatorname{Spec}^\tau S \times_{\operatorname{Spec}^\tau R} (\operatorname{colim}_{[k]\in\Delta^{\mathrm{op}}} \operatorname{Spec} R'^{\times_X k+1})^\tau \\
&\simeq \operatorname{Spec}^\tau S \times_{\operatorname{Spec}^\tau R} X^\tau.
\end{aligned}$$

Here we use that $(-)^\tau$ preserves colimits twice. □

Remark III.3.1.25. If we instead assume that the map $X \to \operatorname{Spec} R$ is flat in Lemma III.3.1.24, then the claim reduces to Lemma III.3.1.23 since flat maps are assumed to be affine, so X must be affine.

Proposition III.3.1.26. *If $p\colon X \to X'$ is a (geometrically) flat map of geometric stacks, then for any geometric stack Y and map $Y \to X'$ the canonical map*

$$(Y \times_{X'} X)^\tau \to Y^\tau \times_{(X')^\tau} X^\tau$$

is an equivalence.

Proof. Suppose first that $X' = \operatorname{Spec} R$ is an affine filtered stack. Let $\operatorname{Spec} A \to Y$ be a flat cover of Y, such that $\operatorname{colim}_{[k]\in\mathbb{\Delta}^{\mathrm{op}}} \operatorname{Spec} A^{\times_Y k+1} \simeq Y$. Furthermore, let $Y' \coloneqq Y \times_{\operatorname{Spec} R} X$ and $Y_A^{[k]} \coloneqq \operatorname{Spec} A^{\times_Y k+1} \times_Y Y'$ such that the squares

$$\begin{tikzcd}
Y_A^{[k]} \arrow[r] \arrow[d, "p_A"] & Y' \arrow[r] \arrow[d, "p'"] & X \arrow[d, "p"] \\
\operatorname{Spec} A^{\times_Y k+1} \arrow[r] & Y \arrow[r] & \operatorname{Spec} R
\end{tikzcd}$$

are cartesian for every $[k] \in \mathbb{\Delta}$. Note that $Y_A \to Y'$ is an effective epimorphism, so we have an equivalence $\operatorname{colim}_{[k]\in\mathbb{\Delta}^{\mathrm{op}}} Y_A^{\times_{Y'} k+1} \simeq Y'$. One can check inductively that

$$Y_A^{\times_{Y'} k+1} \simeq Y_A^{[k]} \simeq \operatorname{Spec} A^{\times_Y k+1} \times_{\operatorname{Spec} R} X.$$

Since $\operatorname{Spec} A \to Y$ is a flat cover, it is in particular affine, hence so is $\operatorname{Spec} A^{\times_Y k+1}$ for every $[k] \in \mathbb{\Delta}$. It follows from Lemma III.3.1.24 that

$$(Y_A^{[k]})^\tau \simeq (\operatorname{Spec} A^{\times_Y k+1} \times_{\operatorname{Spec} R} X)^\tau \simeq (\operatorname{Spec} A^{\times_Y k+1})^\tau \times_{\operatorname{Spec}^\tau R} X^\tau.$$

Combining the above equivalence with the fact that $Y_A \to Y'$ is an effective epimorphism we see that

$$
\begin{aligned}
Y'^{\tau} &\simeq (\operatorname{colim}_{[k]\in\Delta} Y_A^{\times_{Y'}k+1})^{\tau}\\
&\simeq \operatorname{colim}_{[k]\in\Delta} (Y_A^{\times_{Y'}k+1})^{\tau}\\
&\simeq \operatorname{colim}_{[k]\in\Delta} ((\operatorname{Spec} A^{\times_Y k+1})^{\tau} \times_{\operatorname{Spec}^{\tau} R} X^{\tau})\\
&\simeq (\operatorname{colim}_{[k]\in\Delta} \operatorname{Spec} A^{\times_Y k+1})^{\tau} \times_{\operatorname{Spec}^{\tau} R} X^{\tau}\\
&\simeq Y^{\tau} \times_{\operatorname{Spec}^{\tau} R} X^{\tau}.
\end{aligned}
$$

This is what we wanted to prove. The case where X' is any geometric stack remains.

Let $\operatorname{Spec} R \to X'$ be a faithfully flat cover of X'. In this situation $\operatorname{colim}_{[k]\in\Delta^{\mathrm{op}}} \operatorname{Spec} R^{\times_{X'}k+1} \simeq X'$. It follows from the functoriality of slice categories that we get a commutative square

$$
\begin{tikzcd}
\mathrm{Stk}_{/X'} \arrow[r, "\sim"] \arrow[d, "(-)^{\tau}"] & \lim_{[k]\in\Delta} \mathrm{Stk}_{/\operatorname{Spec} R^{\times_{X'}k+1}} \arrow[d, "\phi"] \\
\mathrm{FStack}_{/(X')^{\tau}} \arrow[r, "\sim"] & \lim_{[k]\in\Delta} \mathrm{FStack}_{/(\operatorname{Spec} R^{\times_{X'}k+1})^{\tau}}
\end{tikzcd}
$$

where the horizontal functors are equivalences by [BDL25, Prop.2.1.2.9 (4)]. Here ϕ is the induced functor on the limit of the functors $\mathrm{Stk}_{\operatorname{Spec} R^{\times_{X'}k+1}} \xrightarrow{(-)^{\tau}} \mathrm{FStack}_{(\operatorname{Spec} R^{\times_{X'}k+1})^{\tau}}$ for every $[k] \in \Delta$. Since products in slice categories are given by pullbacks we may reduce to showing that if $X \to X'$ is (geometrically) flat, then $\phi(- \times X) \simeq \phi(-) \times \phi(X)$, which follows immediately from the case where X' is affine. $\square$

Remark III.3.1.27. The structure and proof of Proposition III.3.1.26 follows the proof of [BDL25, Prop. 3.4.3.3] very closely. In fact, this proof is very general and can be used to give criteria for which pullbacks are preserved by a functor $\mathrm{Stk}(\mathcal{C}_0) \to \mathrm{Stk}(\mathcal{C}_1)$ induced by a map of sites $\mathcal{C}_0 \to \mathcal{C}_1$.

Proposition III.3.1.28. *Let X be a geometric spectral stack. If $Y \to X$ is a flat map of spectral stacks, then the map $Y^\tau \to X^\tau$ is a flat map of filtered stacks.*

Proof. Note, that our assumptions imply that Y is a geometric stack, with affine cover given by $Y \times_X \operatorname{Spec} R \to Y$. Here $\operatorname{Spec} R \to X$ is any flat cover of X. Hence, $Y^\tau \to X^\tau$ is a map of geometric filtered stacks. Let S be any connective filtered ring spectrum and let us consider a map $\operatorname{Spec} S \to X^\tau$. We have to show that

1. $Y^\tau \times_{X^\tau} \operatorname{Spec} S$ is an affine filtered stack.

2. The map $\operatorname{Spec} S_Y \simeq Y^\tau \times_{X^\tau} \operatorname{Spec} S \to \operatorname{Spec} S$ induces a flat map $S \to S_Y$ of filtered ring spectra.

Assume first that $X = \operatorname{Spec} R$ is affine. It follows that Y is affine too. Let us denote it as $\operatorname{Spec} R_Y := Y$. Hence $Y^\tau \to X^\tau$ is a map of affine filtered stacks. It follows from a standard Yoneda argument that $Y^\tau \times_{X^\tau} \operatorname{Spec} S$ is affine, which proves (1). We now prove (2). Since $Y \to X$ is flat, we know that $R \to R_Y$ is flat. It follows from Proposition III.2.2.21 that $\tau_{\geqslant \bullet} R \to \tau_{\geqslant \bullet} R_Y$ is flat. Now, since $\operatorname{Spec} S_Y := Y^\tau \times_{X^\tau} \operatorname{Spec} S$ is affine it remains to show that the induced map $S \to S_Y$ is flat, but $S_Y \simeq \tau_{\geqslant \bullet} R_Y \otimes_{\tau_{\geqslant \bullet} R} S$ so the claim follows from Proposition III.2.2.15.

Assume now that X is a geometric stack and that $p\colon \operatorname{Spec} R \to X$ is a faithfully flat cover. It follows from Proposition III.3.1.26 that $p^\tau\colon \operatorname{Spec}^\tau R \to X^\tau$ is an effective epimorphism, since groupoids are effective in FStack. We conclude from Lemma III.3.1.14 that f^τ is flat if and only if the map $Y^\tau \times_{X^\tau} \operatorname{Spec}^\tau R \to \operatorname{Spec}^\tau R$ is flat. We know from Proposition III.3.1.26 that

$$(Y \times_X \operatorname{Spec} R)^\tau \xrightarrow{\sim} Y^\tau \times_{X^\tau} \operatorname{Spec}^\tau R$$

is an equivalence, and that the map diagram

$$\begin{tikzcd} (Y \times_X \operatorname{Spec} R)^\tau \arrow[rr, "\sim"] \arrow[dr] & & Y^\tau \times_{X^\tau} \operatorname{Spec}^\tau R \arrow[dl] \\ & \operatorname{Spec}^\tau R & \end{tikzcd}$$

commutes. Hence we may reduce to the case where X is affine, which we argued above. □

Corollary III.3.1.29. *If X is a geometric spectral stack, then X^τ is a geometric filtered stack.*

Proof. Let $\operatorname{Spec} R \to X$ be a flat cover. It follows from Proposition III.3.1.28 that $\operatorname{Spec}^\tau R \to X^\tau$ is flat and from Proposition III.3.1.26 that it is an effective epimorphism. □

III.3.2 Quasi-coherent sheaves

We now introduce the category of quasi-coherent sheaves on a filtered stack. The theory of quasi-coherent sheaves for filtered stacks is very reminiscent of the theory of quasi-coherent sheaves for spectral stacks. In fact, we will show that if a filtered stack is in the image of $(-)^\tau\colon \mathrm{Stk} \to \mathrm{FStack}$, then $\mathrm{QCoh}(X^\tau)$ is a 1-parameter deformation of $\mathrm{QCoh}(X)$ in the sense of [Bar23].

Construction III.3.2.1. Since $\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0}$ is presentable the functor

$$\mathrm{Mod}_{(-)}(\mathcal{S}p^{\mathrm{fil}})\colon \mathrm{FAff}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$$

given by $\operatorname{Spec} A \mapsto \mathrm{Mod}_A(\mathcal{S}p^{\mathrm{fil}})$. This functor factors through the forgetful functor $\mathrm{CAlg}(\mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L)) \to \mathrm{CAlg}(\mathrm{Pr}^L)$. We will denote functor with target $\mathrm{CAlg}(\mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L))$ by the same name. Since

$\mathrm{Mod}_{(-)}(\mathcal{S}p^{\mathrm{fil}})$ is a sheaf for the flat topology on FAff, it follows that it is a *quasi-coherent context* in the sense [BDL25, Definition 2.1.3.2]. In particular, we obtain a limit preserving functor

$$\mathrm{QCoh}\colon \mathrm{FStack}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L))$$

from [BDL25, Prop. 2.1.2.2].

The very categorical definition of QCoh immediately allows for the following conclusions.

Remark III.3.2.2. Let X be a filtered stack. The quasi-coherent sheaves functor satisfies the following properties:

1. If $X = \operatorname{Spec} A$ for some connective filtered ring spectrum A, then
$$\mathrm{QCoh}(X) = \mathrm{Mod}_A(\mathcal{S}p^{\mathrm{fil}}).$$

2. It is the right Kan extension of
$$\mathrm{Mod}_{(-)}(\mathcal{S}p^{\mathrm{fil}})\colon \mathrm{FAff} \to \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L))$$
along $\operatorname{Spec}\colon \mathrm{FAff} \to \mathrm{FStack}$, which agrees with the right Kan extension of $\mathrm{Mod}_{(-)}(\mathcal{S}p^{\mathrm{fil}})$ postcomposed with either of the forgetful functors
$$\mathrm{CAlg}(\mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L)) \to \mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L) \to \mathrm{Pr}^L.$$
This follows from the fact that both the forgetful functors preserve and create limits.

3. The category $\mathrm{QCoh}(X)$ is a $\mathcal{S}p^{\mathrm{fil}}$-module in Pr^L, where the module structure comes from a symmetric monoidal colimit preserving functor
$$-\otimes \mathcal{O}_X\colon \mathcal{S}p^{\mathrm{fil}} \to \mathrm{QCoh}(X).$$
We denote the image of $\mathbb{S}^{n,k}$ under this functor by $\mathcal{O}_X^{n,k}$.

4. The category $\mathrm{QCoh}(X)$ is a presentably symmetric monoidal stable category. We denote the *monoidal unit* by $\mathcal{O}_X$.

5. If $M, N \in \mathrm{QCoh}(X)$ are quasi-coherent sheaves on X, then there exists a filtered spectrum $\underline{\mathrm{map}}_{\mathrm{QCoh}(X)}(M, N)$ with underlying spectrum $\mathrm{map}_{\mathrm{QCoh}(X)}(M, N)$. If $X = \operatorname{Spec} A$ is affine, then this agrees with the internal hom in $\mathrm{Mod}_A(\mathcal{S}p^{\mathrm{fil}})$. Furthermore, the functor $\underline{\mathrm{map}}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, -)$ is right adjoint to

$$- \otimes \mathcal{O}_X \colon \mathcal{S}p^{\mathrm{fil}} \to \mathrm{QCoh}(X)$$

and we denote it by $\Gamma(X, -) \coloneqq \underline{\mathrm{map}}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, -)$.

6. If $f\colon X \to Y$ is a map of filtered stacks, then there exists a $\mathcal{S}p^{\mathrm{fil}}$-linear symmetric monoidal colimit preserving functor

$$f^*\colon\ \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X),$$

which admits a right adjoint f_*, which is lax $\mathcal{S}p^{\mathrm{fil}}$-linear. In particular, it is lax symmetric monoidal.

7. The f_* functors assemble into a functor

$$\Gamma\colon\ \mathrm{FStack}^{\mathrm{op}} \to \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}),$$

given on a filtered stack Y as follows. The filtered stack $\operatorname{Spec} \mathbb{S}^{0,0}$ is terminal, so there exists a map $f\colon Y \to \operatorname{Spec} \mathbb{S}^{0,0}$. The value of Γ on Y is then given by

$$\Gamma(Y) = f_* \mathcal{O}_Y,$$

which is a filtered ring spectrum since f_* is lax symmetric monoidal.

Definition III.3.2.3. Let X be a filtered stack. We let $\mathrm{QCoh}(X)_{\geqslant 0}$ denote the full subcategory of $\mathrm{QCoh}(X)$ spanned by objects M such that for any connective filtered ring spectrum A and map $p\colon \operatorname{Spec} A \to X$ the object $p^*M \in \mathrm{Mod}_A(\mathcal{S}p^{\mathrm{fil}})$ is connective.

Proposition III.3.2.4. *If X is a filtered stack, then $\mathrm{QCoh}(X)_{\geqslant 0}$ is the connective part of a t-structure on $\mathrm{QCoh}(X)$ such that the following holds:*

0. *If $\operatorname{colim}_{i\in I} \operatorname{Spec} A_i \simeq X$, then the equivalence*

$$\mathrm{QCoh}(X) \simeq \lim_{i\in I^{\mathrm{op}}} \mathrm{Mod}_{A_i}(\mathcal{S}p^{\mathrm{fil}})$$

restricts to an equivalence

$$\mathrm{QCoh}(X)_{\geqslant 0} \simeq \lim_{i\in I^{\mathrm{op}}} \mathrm{Mod}_{A_i}(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0}.$$

In particular $\mathrm{QCoh}(X)_{\geqslant 0}$ is presentable.

1. *If $\mathcal{F}$ is a quasi-coherent sheaf such that for every map $p\colon \operatorname{Spec} A \to X$ the A-module $p^*\mathcal{F}$ is coconnective, then $\mathcal{F}$ is coconnective.*

2. *The t-structure is compatible with the symmetric monoidal structure.*

3. *If $f\colon X \to Y$ is a map of filtered stacks, then f^* is right t-exact, and thus f_* is left t-exact.*

Proof. We first prove (0). Let $\mathrm{QCoh}(X)_{\geqslant 0}$ be as above and choose a presentation $\operatorname{colim}_{i\in I} \operatorname{Spec} A_i \simeq X$. In this situation we have an equivalence

$$\phi\colon\ \mathrm{QCoh}(X) \xrightarrow{\sim} \lim_{i\in I^{\mathrm{op}}} \mathrm{Mod}_{A_i}(\mathcal{S}p^{\mathrm{fil}}).$$

Let $p_i\colon \lim_{i\in I^{\mathrm{op}}} \mathrm{Mod}_{A_i}(\mathcal{S}p^{\mathrm{fil}}) \to \mathrm{Mod}_{A_i}(\mathcal{S}p^{\mathrm{fil}})$ denote the canonical projection for every $i \in I$. We claim that this equivalence restricts to an equivalence

$$\mathrm{QCoh}(X)_{\geqslant 0} \simeq \lim_{i\in I^{\mathrm{op}}} \mathrm{Mod}_{A_i}(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0}.$$

Note, that since A_i is connective for every i and for every $i \to j$ in I the induced map $\mathrm{Mod}_{A_j}(\mathcal{S}p^{\mathrm{fil}}) \to \mathrm{Mod}_{A_i}(\mathcal{S}p^{\mathrm{fil}})$ is right t-exact, the diagram $F\colon I \to \mathrm{FStack} \to \mathrm{Pr}^L$ has a subfunctor with value $\mathrm{Mod}_{A_i}(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0}$ for all $i \in I$. Furthermore, the map

$$\mathrm{QCoh}(X) \to \lim_{i\in I^{\mathrm{op}}} \mathrm{Mod}_{A_i}(\mathcal{S}p^{\mathrm{fil}})$$

sends $\mathrm{QCoh}(X)_{\geqslant 0}$ to $\lim_{i\in I^{\mathrm{op}}} \mathrm{Mod}_{A_i}(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0}$ by definition, so the functor restricts and we conclude that the restriction is fully faithful. It remains to see that the restriction to $\mathrm{QCoh}(X)_{\geqslant 0}$, has essential image $\lim_{i\in I^{\mathrm{op}}} \mathrm{Mod}_{A_i}(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0}$ or equivalently if $p_i\phi M$ is connective for every $i \in I$, then $M \in \mathrm{QCoh}(X)_{\geqslant 0}$. This is clear from the definition of $\mathrm{QCoh}(X)_{\geqslant 0}$.

Now we prove that this forms a t-structure on $\mathrm{QCoh}(X)$. We know from (0), that $\mathrm{QCoh}(X)_{\geqslant 0}$ is presentable so it follows from [Lur17b, Prop. 1.4.4.11] that there is a t-structure on $\mathrm{QCoh}(X)$ with connective part given by $\mathrm{QCoh}(X)_{\geqslant 0}$ if and only if $\mathrm{QCoh}(X)_{\geqslant 0}$ is closed under extensions and colimits in $\mathrm{QCoh}(X)$. This is clear, since for every map $p\colon \operatorname{Spec} R \to X$ the functor p^* is a colimit preserving functor between stable categories and $\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0}$ is closed under extensions and colimits for all connective filtered ring spectra R.

We now prove (1). Suppose M is a quasi-coherent sheaf on X, such that for any map $p\colon \operatorname{Spec} A \to X$ the A-module p^*M is coconnective, then we have to show that for every $N \in \mathrm{QCoh}(X)_{\geqslant 0}$, the mapping space $\mathrm{Map}_{\mathrm{QCoh}(X)}(N, \Omega M)$ is contractible. We have equivalences

$$\begin{aligned} \mathrm{Map}_{\mathrm{QCoh}(X)}(N, \Omega M) &\simeq \lim_{i \in I^{\mathrm{op}}} \mathrm{Map}_{\mathrm{Mod}_{A_i}}(p_i\phi N, p_i\phi\Omega M) \\ &\simeq \lim_{i \in I^{\mathrm{op}}} \mathrm{Map}_{\mathrm{Mod}_{A_i}}(p_i\phi N, \Omega p_i\phi M) \\ &\simeq \lim_{i \in I^{\mathrm{op}}} \mathrm{pt} \\ &\simeq \mathrm{pt}. \end{aligned}$$

Here, the first equivalence follows from ϕ being an equivalence, the second follows from the fact that $p_i\phi$ is an exact functor, the third follows from our assumption that M is coconnective when pulled back to any affine and the fourth equivalence follows the fact that contractible spaces are closed under limits.

For (2), we have to see that $\mathcal{O}_X$ is connective and that if M and N are connective quasi-coherent sheaves on X, then so is $M \otimes N$. Both of which are clear since for every map $p\colon \operatorname{Spec} R \to X$ the functor $p^*\colon \mathrm{QCoh}(X) \to \mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$ is symmetric monoidal and the t-structure on $\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$ is compatible with the symmetric monoidal structure.

Proving (3) remains. Let $f\colon X \to Y$ be a map of filtered stacks and assume that $\mathcal{F}$ is a connective quasi-coherent sheaf on Y. It suffices to see that for every map $p\colon \operatorname{Spec} A \to X$ from an affine filtered stack the A-module $p^*f^*\mathcal{F} \simeq (pf)^*\mathcal{F}$ is connective. This is clear, since $\mathcal{F}$ is connective if and only if it is connective when pulled back to any affine filtered stack. □

Lemma III.3.2.5. *Let $F, G\colon I \to \mathrm{Cat}$ be functors and let $\phi\colon F \to G$ be a natural transformation. If ϕ satisfies the following properties:*

1. *For every $i \in I$ the functor $\phi_i\colon F(i) \to G(i)$ admits a right adjoint ϕ_i^R, which is monadic.*

2. *For every $i \to j \in I$ the square*

$$\begin{tikzcd} F(i) \arrow[r] \arrow[d, "\phi_i"] & F(j) \arrow[d, "\phi_j"] \\ G(i) \arrow[r] & G(j) \end{tikzcd}$$

is right adjointable.

Then the map

$$f\colon \lim_{i\in I} F(i) \to \lim_{i\in I} G(i)$$

admits a right adjoint f^R, and is monadic.

Proof. We aim to apply the Barr–Beck–Lurie theorem [Lur17b, Theorem 4.7.3.5]. So we must verify that

1. The functor f^R is conservative.

2. The functor f^R preserves f^R-split geometric realizations of simplicial objects.

It follows from [Lur17b, Prop. 4.7.4.19] that the diagram

$$\begin{tikzcd} \lim_{i\in I} F(i) \arrow[r, "f_j"] \arrow[d, "f"] & F(j) \arrow[d, "\phi_j"] \\ \lim_{i\in I} G(i) \arrow[r, "g_j"] & G(j) \end{tikzcd}$$

is right adjointable for every j. The functors $f_j\colon \lim_{i\in I} F(i) \to F(j)$ are jointly conservative. We see that f^R is conservative if and only if ϕ_j^R is conservative for every $j \in I$, which follows from our assumption that ϕ_j^R is monadic.

Let $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \lim_{i\in I} G(i)$ be a f^R-split simplicial object. Since the above square is right adjointable, we observe that $g_j X$ is ϕ_j^R-split for every $j \in I$. In particular, its geometric realization is preserved by ϕ_j^R. We want to see that f^R preserves the geometric realization of X. To see this recall that the mapping spaces in a limit are computed as the limit of mapping spaces, and hence we obtain the following string of equivalences:

$$\begin{aligned}
\mathrm{Map}_{\lim_{i\in I} F(i)}(f^R(\mathrm{colim}_{\mathbb{\Delta}^{\mathrm{op}}} X), Y) &\simeq \lim_{i\in I} \mathrm{Map}_{F(i)}(f_i f^R(\mathrm{colim}_{[n]\in\mathbb{\Delta}^{\mathrm{op}}} X_n), f_i Y) \\
&\simeq \lim_{i\in I} \mathrm{Map}_{F(i)}(\phi_i^R g_i(\mathrm{colim}_{\mathbb{\Delta}^{\mathrm{op}}} X), f_i Y) \\
&\simeq \lim_{i\in I} \mathrm{Map}_{F(i)}(\mathrm{colim}_{\mathbb{\Delta}^{\mathrm{op}}} \phi_i^R g_i(X), f_i Y) \\
&\simeq \lim_{i\in I} \lim_{[n]\in\mathbb{\Delta}} \mathrm{Map}_{F(i)}(\phi_i^R g_i(X_n), f_i Y) \\
&\simeq \lim_{[n]\in\mathbb{\Delta}} \lim_{i\in I} \mathrm{Map}_{F(i)}(\phi_i^R g_i(X_n), f_i Y) \\
&\simeq \lim_{[n]\in\mathbb{\Delta}} \mathrm{Map}_{\lim_{i\in I} F(i)}(\phi_i^R g_i(X_n), f_i Y) \\
&\simeq \lim_{[n]\in\mathbb{\Delta}} \mathrm{Map}_{\lim_{i\in I} F(i)}(f_i f^R X, f_i Y) \\
&\simeq \mathrm{Map}_{\lim_{i\in I} F(i)}(\mathrm{colim}_{[n]\in\mathbb{\Delta}^{\mathrm{op}}} f^R X_n, Y).
\end{aligned}$$

This completes the proof. □

Proposition III.3.2.6. *If $p\colon X \to Y$ is an affine morphism of filtered stacks, then the functor*

$$p^*\colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$$

is comonadic. That is, in the commutative triangle

$$\begin{tikzcd}
\mathrm{QCoh}(Y) \arrow[rr, "p_*\mathcal{O}_X \otimes -"] \arrow[dr, "p^*"'] & & \mathrm{Mod}_{p_*\mathcal{O}_X}(\mathrm{QCoh}(Y)) \arrow[dl, "\tilde{p}^*"] \\
& \mathrm{QCoh}(X) &
\end{tikzcd}$$

the map $\tilde{p}^*$, induced by p^*, is an equivalence.

Proof. Let $p\colon X \to Y$ be an affine map of filtered stacks. Consider $\operatorname{colim}_{i\in I} \operatorname{Spec} R_i \xrightarrow{\sim} Y$ a presentation of Y. In this situation, the fiber product $\operatorname{Spec} R_i \times_Y X \simeq \operatorname{Spec} R_i'$ is affine for every $i \in I$. Furthermore, since colimits are universal in FStack we obtain an equivalence:

$$\begin{aligned}\operatorname{colim}_{i\in I} \operatorname{Spec} R_i' &\simeq \operatorname{colim}_{i\in I} \operatorname{Spec} R_i \times_Y X \\ &\simeq (\operatorname{colim}_{i\in I} \operatorname{Spec} R_i) \times_Y X \\ &\simeq X.\end{aligned}$$

Applying $\operatorname{QCoh}(-)$, we see that the horizontal maps in

$$\begin{tikzcd} \operatorname{QCoh}(Y) \arrow[r, "\sim"] \arrow[d, "p^*"] & \lim_{i\in I^{\mathrm{op}}} \operatorname{Mod}_{R_i}(\mathcal{S}p^{\mathrm{fil}}) \arrow[d] \\ \operatorname{QCoh}(X) \arrow[r, "\sim"] & \lim_{i\in I^{\mathrm{op}}} \operatorname{Mod}_{R_i'}(\mathcal{S}p^{\mathrm{fil}}) \end{tikzcd}$$

are equivalences. Therefore, we may reduce to showing that

$$f\colon \lim_{i\in I^{\mathrm{op}}} \operatorname{Mod}_{R_i}(\mathcal{S}p^{\mathrm{fil}}) \to \lim_{i\in I^{\mathrm{op}}} \operatorname{Mod}_{R_i'}(\mathcal{S}p^{\mathrm{fil}})$$

is comonadic, which follows from Lemma III.3.2.5 and the fact that the forgetful functor $\operatorname{Mod}_B(\mathcal{S}p^{\mathrm{fil}}) \to \operatorname{Mod}_A(\mathcal{S}p^{\mathrm{fil}})$ is monadic for every map of filtered ring spectra $A \to B$. $\square$

Corollary III.3.2.7. *If $f\colon X \to Y$ is an affine map of filtered stacks, then the functor $f_*\colon \operatorname{QCoh}(X) \to \operatorname{QCoh}(Y)$ admits a right adjoint.*

Corollary III.3.2.8 ([BDL25, Theorem 2.2.3.1.])**.** *If X is a filtered stack, then the functor*

$$\mathrm{FStack}^{\mathrm{aff}}_{/X} \to \operatorname{CAlg}(\operatorname{QCoh}(X))^{\mathrm{op}}$$

sending an affine morphism $f\colon Y \to X$ to $f_*\mathcal{O}_Y$ is an equivalence.

Proof. The functor $\mathrm{FStack}^{\mathrm{op}} \to \mathrm{Cat}$ sending X to $\mathrm{FStack}_{/X}$ preserves limits by Lemma III.3.1.14. Thus, we may assume that $X = \operatorname{Spec} R$ is affine, where the result follows from the well-known equivalence

$$\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})^{\mathrm{op}}_{R/} \xrightarrow{\sim} \mathrm{CAlg}(\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}}))^{\mathrm{op}}.$$

$\square$

Definition III.3.2.9. Let X be a filtered stack and let

$$\underline{\mathrm{Spec}}_X(-)\colon\ \mathrm{CAlg}(\mathrm{QCoh}(X))^{\mathrm{op}} \to \mathrm{FStack}^{\mathrm{aff}}_{/X}$$

denote the inverse of the functor in Corollary III.3.2.8. If $A \in \mathrm{CAlg}(\mathrm{QCoh}(X))$ is a quasi-coherent commutative algebra over X, then *the relative spectrum* of A over X is the stack $\underline{\mathrm{Spec}}_X(A)$.

Proposition III.3.2.10. *If the square*

$$\begin{tikzcd} X' \arrow[r, "h'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

is cartesian and f is affine, then the square

$$\begin{tikzcd} \mathrm{QCoh}(Y) \arrow[r, "h'^*"] \arrow[d, "f^*"] & \mathrm{QCoh}(Y') \arrow[d, "f'^*"] \\ \mathrm{QCoh}(X) \arrow[r, "h^*"] & \mathrm{QCoh}(X') \end{tikzcd}$$

is right adjointable. That is the Beck–Chevalley map

$$h'^* f_* \to f'_* h^*$$

is an equivalence.

Proof. By Proposition III.3.2.6 the map $f\colon X \to Y$ is universally 0-affine. Hence, so is $f'\colon X' \to Y'$, since affine maps are closed under pull-back. In this situation the claim was proven in [BDL25, Prop. 2.2.2.5 (1)]. $\square$

Proposition III.3.2.11. *If $f\colon Y \to X$ is an affine map of filtered stacks, then*

$$f_*\colon \operatorname{QCoh}(Y) \to \operatorname{QCoh}(X)$$

is t-exact.

Proof. Since f_* is left t-exact by Proposition III.3.2.4 it suffices to show that for every $\mathcal{F} \in \operatorname{QCoh}(Y)^{\heartsuit}$ the quasi-coherent sheaf $f_*\mathcal{F}$ is discrete. Furthermore, we know that it is coconnective, so it suffices to see that $f_*\mathcal{F}$ is connective. Let $p\colon \operatorname{Spec} R \to X$ be any map and let $h\colon \operatorname{Spec} S \simeq \operatorname{Spec} R \times_X Y$ and $q\colon \operatorname{Spec} S \to Y$ be the maps induced by pulling back along f. By Proposition III.3.2.10 we have an equivalence $p^* f_*\mathcal{F} \simeq h_* q^*\mathcal{F}$. Since $\mathcal{F}$ was discrete, $q^*\mathcal{F}$ is connective, and hence so is $h_* q^*\mathcal{F}$ by Proposition III.2.1.26. Therefore, $f_*\mathcal{F}$ is connective by Proposition III.3.2.4. $\square$

Theorem III.3.2.12. If X is a geometric filtered stack, then the canonical t-structure on $\operatorname{QCoh}(X)$ satisfies the following properties.

1. If $M \in \operatorname{QCoh}(X)$ is a quasi-coherent sheaf on X, then the following are equivalent
   (a) The quasi-coherent sheaf M is coconnective.
   (b) If $p\colon \operatorname{Spec} R \to X$ is a flat cover, then p^*M is coconnective.
2. The t-structure is left and right complete.
3. The t-structure is compatible with filtered colimits.
4. There exists a functor $\operatorname{QCoh}(X)^{\heartsuit} \to \operatorname{QCoh}(X^{\heartsuit})^{\heartsuit}$.

Proof. Fix a flat cover $p\colon \operatorname{Spec} R \to X$ and let $\operatorname{Spec} R^\bullet$ denote the Čech nerve of p. We start by proving (1). It follows from part (0) of Proposition III.3.2.4 that

$$\mathrm{QCoh}(X)_{\geqslant 0} \xrightarrow{f^*} \mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0} \rightrightarrows \mathrm{Mod}_{R^1}(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0} \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \mathrm{Mod}_{R^2}(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0} \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} \cdots$$

is a limit diagram in Pr^L. Moreover, the transition maps are left exact, by the assumption that p^* is a flat cover and Proposition III.2.2.13. It follows from [Lurb, Prop.C.2.3.4.] this is also a limit diagram in $\mathrm{Groth}^{\mathrm{lex}}$, the category of Grothendieck prestable categories with left exact functors. So $f^*\colon \mathrm{QCoh}(X)_{\geqslant 0} \to \mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0}$ is left exact or equivalently $f^*\colon \mathrm{QCoh}(X) \to \mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$ is t-exact.

One can also check that f^* is conservative. This can be seen as follows. We have that $f^* \simeq p_0 p$, where

$$p\colon \mathrm{QCoh}(X) \to \lim_{[n]\in\Delta_s} \mathrm{Mod}_{R^n}(\mathcal{S}p^{\mathrm{fil}})$$

is the chosen equivalence, and $p_0\colon \lim_{[n]\in\Delta_s} \mathrm{Mod}_{R^n}(\mathcal{S}p^{\mathrm{fil}}) \to \mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$ is the canonical projection. Now, $[0] \in \Delta_s$ is initial, so the diagram

$$\begin{tikzcd}
 & \lim_{[n]\in\Delta_s} \mathrm{Mod}_{R^n}(\mathcal{S}p^{\mathrm{fil}}) \arrow[dl, "p_0"'] \arrow[dr, "p_k"] & \\
\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}}) \arrow[rr] & & \mathrm{Mod}_{R^n}(\mathcal{S}p^{\mathrm{fil}})
\end{tikzcd}$$

commutes for every n. In particular, if $f^*\mathcal{F} \simeq 0$, then $p_n p(\mathcal{F}) \simeq 0$ for every n. Now, the canonical functor

$$\lim_{[n]\in\Delta_s} \mathrm{Mod}_{R^n}(\mathcal{S}p^{\mathrm{fil}}) \to \prod_{[n]\in\Delta_s} \mathrm{Mod}_{R^n}(\mathcal{S}p^{\mathrm{fil}}),$$

induced by the collection $\{p_n\}_{[n]\in\Delta_s}$, is conservative by [HP25, Lemma B.1]. Hence, $\mathcal{F} \simeq 0$.

Since f^* is t-exact and conservative an object is (co)connective if and only if its image under f^* is (co)connective.

We now prove (2). We first prove that the t-structure is right complete. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. We have to show that $\operatorname{colim}_k \tau_{\geqslant k}\mathcal{F} \to \mathcal{F}$ is an equivalence. Since f^* is conservative it suffices to see that

$$f^*(\operatorname{colim}_k \tau_{\geqslant k}\mathcal{F}) \to f^*(\mathcal{F})$$

is an equivalence. Now, f^* preserves colimits and is t-exact by (1). So the conclusion follows from Proposition III.2.1.26. The argument to see that the t-structure is left complete is analogous, if we can show that

$$f^*(\lim_k \tau_{\leqslant k}\mathcal{F}) \to \lim_k f^*(\tau_{\leqslant k}\mathcal{F})$$

is an equivalence. To see this we consider the Milnor $\lim^1$-sequence

$$0 \to \lim_k{}^1 \pi^P_{n+1}(f^*(\tau_{\leqslant k}\mathcal{F})) \to \pi^P_n(\lim_k f^*(\tau_{\leqslant k}\mathcal{F})) \to \lim_k \pi^P_n(f^*(\tau_{\leqslant k}\mathcal{F})) \to 0.$$

Since f^* is t-exact, the system $\{\pi^P_{n+1}(f^*(\tau_{\leqslant k}\mathcal{F}))\}_{k\to\infty}$ is Mittag–Leffler for every n. Hence, the map

$$\pi^P_n(\lim_k f^*(\tau_{\leqslant k}\mathcal{F})) \to \lim_k \pi^P_n(f^*(\tau_{\leqslant k}\mathcal{F})) \simeq \lim_k f^*(\pi^P_n(\tau_{\leqslant k}\mathcal{F})) \simeq \pi^P_n(f^*(\mathcal{F}))$$

is an equivalence for every n. We conclude that $f^*(\lim_k \tau_{\leqslant k}\mathcal{F}) \to \lim_k f^*(\tau_{\leqslant k}\mathcal{F})$ is an equivalence, since the Postnikov t-structure on $\operatorname{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$ is separated.

We now prove (3). Let $F\colon I \to \operatorname{QCoh}(X)$ be a filtered diagram which factors through $\operatorname{QCoh}(X)^\heartsuit$. We have to show that $\operatorname{colim}_{i\in I} F(i)$ is discrete. Since f^* is t-exact and conservative, it suffices to see that $\operatorname{colim}_{i\in I} f^*(F(i)) \simeq f^*(\operatorname{colim}_{i\in I} F(i))$ is discrete. Now, since f^* is t-exact the R-module $f^*(F(i))$ is discrete for every $i \in I$. The Postnikov t-structure was seen to be compatible with filtered colimits in Proposition III.2.1.26, so $\operatorname{colim} f^*(F(i))$ is discrete, which completes the proof.

Finally, we prove (4). It follows from (1) and [Lurb, Prop. C.5.4.5] that

$$\mathrm{QCoh}(X)^{\heartsuit} \simeq \lim_{[k]\in\Delta} \mathrm{Mod}_{R^k}(\mathcal{S}p^{\mathrm{fil}})^{\heartsuit}.$$

A similar argument shows that

$$\mathrm{QCoh}(X^{\heartsuit})^{\heartsuit} \simeq \lim_{[k]\in\Delta} \mathrm{Mod}_{\pi_0^P R^k}(\mathrm{Ab}^{\mathrm{gr}}).$$

It follows from Proposition III.2.1.26 that there exists an equivalence $\mathrm{Mod}_{R^k}(\mathcal{S}p^{\mathrm{fil}})^{\heartsuit} \simeq \mathrm{Mod}_{\pi_0^P R^k}(\mathrm{Ab}^{\mathrm{gr}})$. If we combine the above equivalences, then we obtain the claim. All of these equivalences are symmetric monoidal. □

Remark III.3.2.13. In the proof of (4) above, one could have constructed a generalized Eilenberg–Maclane functor

$$D(\mathrm{Mod}_{\pi_0^P R^k}(\mathrm{Ab}^{\mathrm{gr}}))_{\geqslant 0} \to \mathrm{Mod}_{R^k}(\mathcal{S}p^{\mathrm{fil}})_{\geqslant 0},$$

which induces an equivalence on the heart. This would have allowed for a functorial comparison of hearts of the two categories. We decided not to use this strategy, since we do not need this extra functoriality for this paper.

For the final part of this section we will focus on etale morphisms of filtered stacks, and show that a version of etale rigidity extends to the class of geometric filtered stacks coming from geometric spectral stacks.

Definition III.3.2.14. A map $f\colon Y \to X$ of filtered stacks is *etale* if it is affine and for every map $\operatorname{Spec} R \to X$ the map $\operatorname{Spec} S \simeq \operatorname{Spec} R \times_X Y \to \operatorname{Spec} R$ is defined via an etale map $R \to S$ of filtered commutative ring spectra.

Remark III.3.2.15. It follows immediately from the definition that etale maps of filtered stacks are flat.

Remark III.3.2.16. Similarly, we define a map of spectral stacks $X \to Y$ to be etale if it is affine and the base-change to any affine is π_*-etale.

Proposition III.3.2.17. *If*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "p'"] & Y \arrow[d, "p"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

is a cartesian square of filtered stacks and $p\colon Y \to X$ is an effective epimorphism, then f is etale if and only if f' is.

Proof sketch. It is clear that if f is etale, then so is f'. So we focus on the converse. The same argument as in Lemma III.3.1.14 allows us to reduce to the following claim: Let $R \to S$ be a faithfully flat map of filtered ring, in this situation $R \to R'$ is etale if and only if $S \to R' \otimes_R S$ is etale. By Corollary III.2.4.8 we know that $R \to R'$ is flat if and only if $S \to R' \otimes_R S$ is flat, so we can reduce to the analogous statement for Dirac rings, which we expect will appear in future work of Hesselholt–Pstrągowski. $\square$

Warning III.3.2.18. This proof relies on a result that we expect to appear in future work of Hesselholt–Pstrągowski. For this reason we have denoted the above argument as a proof sketch. The following results are reliant on this result and as a consequence, we encourage the reader to think of the results as conjectural or contingent on the relevant statements.

Theorem III.3.2.19. If X is a geometric spectral stack, then the functors

$$\mathrm{Stk}_{/X} \xrightarrow{(-)^{\tau}} \mathrm{FStack}_{/X^{\tau}} \xrightarrow{(-)^{\heartsuit}} \mathrm{DStk}_{/X^{d\heartsuit}}$$

restricts to an equivalence on the full subcategories $\mathrm{Stk}^{\mathrm{et}}_{/X}, \mathrm{FStack}^{\mathrm{et}}_{/X^{\tau}}$, and $\mathrm{DStk}^{\mathrm{et}}_{/X^{d\heartsuit}}$ spanned by spectral (respectively filtered, respectively Dirac) etale stacks over X (respectively X^{τ}, respectively $X^{d\heartsuit}$).

Proof. Let $p\colon \operatorname{Spec} R \to X$ be a flat cover of X. It follows from Corollary III.3.1.29 and the analogous statement for $(-)^{\heartsuit}$ that X^{τ} and $X^{d\heartsuit}$ are geometric filtered and Dirac stacks respectively. In particular, if $\operatorname{Spec} R^{\bullet} \to \operatorname{Spec} R$ denotes the Čech nerve of p, then we obtain equivalences

$$\mathrm{Stk}_{/X} \simeq \lim_{[n]\in\Delta} \mathrm{Stk}_{/\operatorname{Spec} R^n}, \quad \mathrm{FStack}_{/X^{\tau}} \simeq \lim_{[n]\in\Delta} \mathrm{FStack}_{/\operatorname{Spec}^{\tau} R^n},$$

and

$$\mathrm{DStk}_{/X^{d\heartsuit}} \simeq \lim_{[n]\in\Delta} \mathrm{DStk}_{/\operatorname{Spec} \pi_*(R^n)}.$$

The proof comes into steps

1. Show that the above equivalences restrict to the full subcategories of etale stacks.

2. Apply etale rigidity, Theorem III.2.2.30, for commutative algebras.

We first prove (1). Here we will only show the case of filtered stacks the other two are analogous. Since etale maps are closed under base-change we see that the functor restricts to a fully faithful functor

$$\mathrm{FStack}^{\mathrm{et}}_{/X^{\tau}} \hookrightarrow \lim_{[n]\in\Delta} \mathrm{FStack}_{/\operatorname{Spec}^{\tau} R^n}.$$

This functor is essentially surjective by Proposition III.3.2.17.

We now prove (2). By (1), we may assume that X is affine, so we are reduced to showing that the functors

$$\mathrm{Stk}^{\mathrm{et}}_{/\operatorname{Spec} R} \to \mathrm{FStack}^{\mathrm{et}}_{/\operatorname{Spec}^{\tau} R} \to \mathrm{DStk}^{\mathrm{et}}_{/\operatorname{Spec} \pi_* R},$$

are equivalences, but this is clear, since every etale morphism is affine, and hence this reduces to Theorem III.2.2.30 by Corollary III.3.2.8 and its spectral and Dirac variants. □

Remark III.3.2.20. In the setting of MU-based synthetic spectra, Robert Burklund has proposed an analogous argument in order to construct Morava E-theories, by lifting their MU-homology, which are p-completed etale algebras in $D(\mathrm{Comod}_{\mathrm{MU}_*\mathrm{MU}})$ the derived category of $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$-comodules.

III.3.3 Deformations of QCoh from filtered geometry

Now that we have established a well-functioning theory of quasi-coherent sheaves on a filtered stack we want to connect this to the theory of "synthetic deformations of stable categories", which has served as a strong tool in stable homotopy theory in the last couple of years. In this section we see that if X is a spectral stack, then quasi-coherent sheaves on X^{τ} are a 1-parameter deformation of $\mathrm{QCoh}(X)$, with special fiber given by $\mathrm{QCoh}(X^{\heartsuit})$. Furthermore, we study the case where X is geometric, which allows for a finer analysis of the situation.

Definition III.3.3.1 ([Bar23, Definition 2.3])**.** If $\mathcal{C} \in \mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L)$ is a $\mathcal{S}p^{\mathrm{fil}}$-linear presentable stable category, then its *generic fiber* is given by the relative tensor product:

$$\mathcal{C}[\tau^{-1}] := \mathcal{C} \underset{\mathcal{S}p^{\mathrm{fil}}}{\otimes} \mathcal{S}p.$$

Here the category of spectra is given a $\mathcal{S}p^{\mathrm{fil}}$-linear structure via the symmetric monoidal functor $\mathcal{S}p^{\mathrm{fil}} \xrightarrow{\tau^{-1}} \mathcal{S}p$.

Furthermore, the *special fiber* of $\mathcal{C}$ is defined as the relative tensor product:

$$\mathrm{gr}_* \mathcal{C} := \mathcal{C} \underset{\mathcal{S}p^{\mathrm{fil}}}{\otimes} \mathcal{S}p^{\mathrm{gr}}.$$

The $\mathcal{S}p^{\mathrm{fil}}$-linear structure on $\mathcal{S}p^{\mathrm{gr}}$ is induced by the symmetric monoidal functor $\mathrm{gr}_* \mathcal{S}p^{\mathrm{fil}} \to \mathcal{S}p^{\mathrm{gr}}$.

Definition III.3.3.2. If $\mathcal{C} \in \mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L)$ is $\mathcal{S}p^{\mathrm{fil}}$-linear, then the functors $\tau^{-1} \colon \mathcal{S}p^{\mathrm{fil}} \to \mathcal{S}p$ and $\mathrm{gr}_* \colon \mathcal{S}p^{\mathrm{fil}} \to \mathcal{S}p^{\mathrm{gr}}$ induce functors

$$\mathcal{C} \to \mathcal{C}[\tau^{-1}] \quad \text{and} \quad \mathcal{C} \to \mathrm{gr}_* \mathcal{C}.$$

We denote these by τ^{-1} and gr_* respectively.

Example III.3.3.3. If R is a filtered ring spectrum, then the generic fiber of $\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{fil}})$ is given by $\mathrm{Mod}_{R[\tau^{-1}]}(\mathcal{S}p)$ and the special fiber is given by $\mathrm{Mod}_{\mathrm{gr}_* R}(\mathcal{S}p^{\mathrm{gr}})$. In particular, if $R = \tau_{\geqslant \bullet} S$ for some ring spectrum S, then we see that the generic fiber is $\mathrm{Mod}_S(\mathcal{S}p)$ and the special fiber is given by $\mathrm{Mod}_{\Sigma^* \pi_* S}(\mathcal{S}p^{\mathrm{gr}}) = D(\mathrm{Mod}_{\pi_* S}(\mathrm{Ab}^{\mathrm{gr}}))$.

Recall the following result, from Section III.2.5.

Proposition III.3.3.4 ([Bar23, Lemma 2.23.])**.** *The functors*

$$\mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L) \xrightarrow{(-)[\tau^{-1}]} \mathrm{Pr}^L_{\mathrm{st}}$$

and

$$\mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L) \xrightarrow{\mathrm{gr}_*} \mathrm{Mod}_{\mathcal{S}p^{\mathrm{gr}}}(\mathrm{Pr}^L)$$

preserve limits.

The proof of Proposition III.3.3.4 is simple. It is proven directly that $\mathcal{S}p$ and $\mathcal{S}p^{\mathrm{gr}}$ are dualizable as $\mathcal{S}p^{\mathrm{fil}}$-linear categories, which implies the result. One might believe showing that these categories are dualizable in $\mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L)$ is difficult, but this turns out to be wrong. This is a consequence of the extensive work on the topic of dualizability in Pr^L due to Maxime Ramzi and Sasha Efimov in recent years cf. [Ram24] and turn out to have a very robust theory.

Proposition III.3.3.5. *If X is a filtered stack with a presentation $\mathrm{colim}_{i\in I}\operatorname{Spec} R_i \xrightarrow{\sim} X$, then the generic fiber of $\mathrm{QCoh}(X)$ is given by*

$$\mathrm{QCoh}(X)[\tau^{-1}] \simeq \lim_{i\in I^{\mathrm{op}}} \mathrm{Mod}_{R_i[\tau^{-1}]}(\mathcal{S}p),$$

and the special fiber is given by

$$\mathrm{gr}_*\mathrm{QCoh}(X) \simeq \lim_{i\in I^{\mathrm{op}}} \mathrm{Mod}_{\pi_0^P R_i}(\mathcal{S}p^{\mathrm{gr}}) \simeq \mathrm{QCoh}(X^{\heartsuit}).$$

In particular, if $X = Y^\tau$ for some spectral stack Y, then we obtain equivalences

$$\mathrm{QCoh}(Y^\tau)[\tau^{-1}] \xrightarrow{\sim} \mathrm{QCoh}(Y) \quad \textit{and} \quad \mathrm{gr}_*\mathrm{QCoh}(Y^\tau) \xrightarrow{\sim} \mathrm{QCoh}(Y^{d\heartsuit}).$$

Proof. This follows immediately from Proposition III.3.3.4 and the fact that there is an equivalence

$$\mathrm{QCoh}(X) \xrightarrow{\sim} \lim_{i\in I^{\mathrm{op}}} \mathrm{Mod}_{R_i}(\mathcal{S}p^{\mathrm{fil}}).$$

The case where $X = Y^\tau$ for some spectral stack Y, follows from the above and Example III.3.3.3. □

Remark III.3.3.6. The above result already tells us that for a spectral stack X the category $\mathrm{QCoh}(X^\tau)$, carries a lot of information about the stack X and its category of quasi-coherent sheaves. However, one tool which is readily available in the case of synthetic spectra is still missing. Namely, the *synthetic analogue* $\nu\colon \mathcal{S}p \to \mathrm{Syn}_E$. The synthetic analogue is lax symmetric monoidal, but does not preserve limits or colimits in general. In fact, for an exact sequence of spectra $X \to Y \to Z$ the image $\nu X \to \nu Y \to \nu Z$ is exact if and only if the induced map on $E_*Y \to E_*Z$ is surjective. As a consequence we can not expect to be able to construct a synthetic analogue $\nu\colon\ \mathrm{QCoh}(X) \to \mathrm{QCoh}(X^\tau)$ using the methods above.

The rest of this section will be spent constructing such a functor in the case of geometric spectral stacks and studying its properties.

Lemma III.3.3.7. *If $F\colon \mathcal{C} \to \mathcal{D}$ is a lax symmetric monoidal functor of symmetric monoidal categories which admit geometric realizations, then it defines a lax natural transformation of functors*

$$F^\sharp\colon\ \mathrm{Mod}_-(\mathcal{C}) \Rightarrow \mathrm{Mod}_{F(-)}(\mathcal{D}),$$

when we consider $\mathrm{Mod}_-(\mathcal{C})$ *and* $\mathrm{Mod}_{F(-)}(\mathcal{D})$ *as functors* $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Cat}$.

Furthermore, if $\mathrm{CAlg}(\mathcal{C})^{F^\flat}$ *denotes the wide subcategory of* $\mathrm{CAlg}(\mathcal{C})$ *spanned by maps $A \to B$, for which the natural transformation*

$$F(B) \underset{F(A)}{\otimes} F(-) \to F(B \underset{A}{\otimes} -)$$

is an equivalence, then $F^\sharp$ is a natural transformation when restricted to $\mathrm{CAlg}(\mathcal{C})^{F^\flat}$.

Proof. For any symmetric monoidal category $\mathcal{C}$, which admits colimits of simplicial objects and where the tensor product preserves such geometric realizations separately in each variable, the lax symmetric monoidal functor

$$\mathrm{Alg}(\mathcal{C}) \to \mathrm{Cat}$$

sending an associative algebra A to $\mathrm{LMod}_A(\mathcal{C})$ is induced from a map of operads $\mathrm{LMod}(\mathcal{C})^{\otimes} \to \mathrm{Alg}(\mathcal{C})^{\otimes}$ via straightening. Specifically, these categories are given by the internal hom for the Boardman–Vogt tensor product from LM and Assoc respectively, and the map is induced by the forgetful functor $\mathrm{LM} \to \mathrm{Assoc}$. This map is a cocartesian fibration by the assumptions we made on $\mathcal{C}^{\otimes}$. Dunn additivity[1] allows us to define a functor

$$\mathrm{CAlg}(\mathcal{C}) \times \mathrm{Fin}_* \to \mathrm{Alg}(\mathcal{C})^{\otimes}$$

refining the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$. Consider the fiber product

$$\mathrm{Mod}(\mathcal{C})^{\otimes} := (\mathrm{CAlg}(\mathcal{C}) \times \mathrm{Fin}_*) \times_{\mathrm{Alg}(\mathcal{C}^{\otimes})} \mathrm{LMod}(\mathcal{C})^{\otimes}.$$

Since $\mathrm{LMod}(\mathcal{C})^{\otimes} \to \mathrm{Alg}(\mathcal{C})^{\otimes}$ was a cocartesian fibration, so is the map $\mathrm{Mod}(\mathcal{C})^{\otimes} \to \mathrm{CAlg}(\mathcal{C}) \times \mathrm{Fin}_*$. The fiber over any commutative algebra $A \in \mathrm{CAlg}(\mathcal{C})$ is exactly the symmetric monoidal category $\mathrm{Mod}_A(\mathcal{C})^{\otimes} \to \mathrm{Fin}_*$. Repeating this construction for $\mathcal{D}$ and using the bifunctoriality of the Boardman–Vogt tensor product (and its internal hom), we obtain a commutative diagram:

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C})^{\otimes} \arrow[r, "F"] \arrow[d] & \mathrm{Mod}(\mathcal{D})^{\otimes} \arrow[d] \\
\mathrm{CAlg}(\mathcal{C}) \times \mathrm{Fin}_* \arrow[r, "F"] \arrow[d] & \mathrm{CAlg}(\mathcal{D}) \arrow[d] \\
\mathrm{CAlg}(\mathcal{C}) \arrow[r, "F"] & \mathrm{CAlg}(\mathcal{D}).
\end{tikzcd}$$

Here the verticals maps are cocartesian fibrations by construction. Thus the pullback $F^* \mathrm{Mod}(\mathcal{D})^{\otimes} \to \mathrm{CAlg}(\mathcal{C})$ of $\mathrm{Mod}(\mathcal{D})^{\otimes} \to \mathrm{CAlg}(\mathcal{D})$ along

[1]Specifically the identification $\mathrm{Comm} \otimes_{\mathrm{BV}} \mathrm{Assoc} \simeq \mathrm{Comm}$.

$F\colon \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{D})$ is also a cocartesian fibration the straightening. Furthermore, the functors $\mathrm{Mod}_{(-)}(\mathcal{C}), \mathrm{Mod}_{F(-)}(\mathcal{D})\colon \mathrm{CAlg}(\mathcal{C}) \to \mathrm{Cat}_{/\mathrm{Fin}_*}$ obtained via straigthening are both take values in the subcategory of symmetric monoidal categories and symmetric monoidal functors.

By our assumption that $F\colon \mathcal{C} \to \mathcal{D}$ is lax symmetric monoidal or equivalently $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a map of operads, we conclude that the induced map $F^{\sharp}\colon \mathrm{Mod}(\mathcal{C})^{\otimes} \to F^* \mathrm{Mod}(\mathcal{D})^{\otimes}$ straightens to a lax natural transformation

$$F^{\sharp}\colon \mathrm{Mod}_{(-)}(\mathcal{C}) \Rightarrow \mathrm{Mod}_{F(-)}(\mathcal{D}).$$

It is clear that $F^{\sharp}$ restricts to a natural transformation on $\mathrm{CAlg}(\mathcal{C})^{F^{\flat}}$. □

Proposition III.3.3.8. *If X is a geometric spectral stack and suppose $p\colon \operatorname{Spec} R \to X$ is a flat cover, then there exists a lax symmetric monoidal functor*

$$\nu\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(X^{\tau}),$$

such that the diagram

$$\begin{tikzcd}
\mathrm{QCoh}(X) \arrow[r, "\nu"] \arrow[d, "p^*"] & \mathrm{QCoh}(X^{\tau}) \arrow[d, "(p^{\tau})^*"] \\
\mathrm{Mod}_R(\mathcal{S}p) \arrow[r, "\tau_{\geqslant \bullet}"] & \mathrm{Mod}_{\tau_{\geqslant \bullet} R}(\mathcal{S}p^{\mathrm{fil}})
\end{tikzcd}$$

commutes.

Proof. Let $\operatorname{Spec} R^*$ denote the Čech nerve of p. In this situation, we have equivalences $\mathrm{QCoh}(X) \simeq \lim_{[n]\in\Delta} \mathrm{Mod}_{R^n}(\mathcal{S}p)$ and $\mathrm{QCoh}(X^{\tau}) \simeq \lim_{[n]\in\Delta} \mathrm{Mod}_{\tau_{\geqslant \bullet} R^n}(\mathcal{S}p^{\mathrm{fil}})$. In particular, there exists a diagram $R^*\colon \Delta \to$

$\mathrm{CAlg}(\mathcal{S}p)$ such that composing with $\mathrm{Mod}_{-}(\mathcal{S}p)$ and $\mathrm{Mod}_{\tau_{\geqslant \bullet}(-)}(\mathcal{S}p^{\mathrm{fil}})$ respectively defines the diagrams whose limit is $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(X^{\tau})$ respectively.

The (non-full) inclusion $\mathbb{\Delta}_s \subseteq \mathbb{\Delta}$ is initial [Lur09b, Lemma 6.2.3.7], so we may restrict the diagram to $\mathbb{\Delta}_s$. In this situation, the diagram takes values in the wide subcategory $\mathrm{CAlg}(\mathcal{S}p)^{\flat}$ spanned by flat morphisms. If we let $\mathrm{CAlg}(\mathcal{S}p)^{(\tau_{\geqslant \bullet})^{\flat}}$ be defined as in Lemma III.3.3.7, then it follows from Lemma III.2.2.20 that we have an inclusion $\mathrm{CAlg}(\mathcal{S}p)^{\flat} \subseteq \mathrm{CAlg}(\mathcal{S}p)^{(\tau_{\geqslant \bullet})^{\flat}}$. As a consequence, Lemma III.3.3.7 provides a natural transformation

$$\mathrm{Mod}_{R^*}(\mathcal{S}p) \Rightarrow \mathrm{Mod}_{(\tau_{\geqslant \bullet} R)^*}(\mathcal{S}p^{\mathrm{fil}})$$

of functors $\mathbb{\Delta}_s \to \mathrm{CAlg}(\mathrm{Cat})^{\mathrm{lax}}$ with lax symmetric monoidal components. Taking the limits of this natural transformation of diagram induces a lax symmetric monoidal functor

$$\nu\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(X^{\tau}). \qquad \square$$

Definition III.3.3.9. Let X be a geometric spectral stack. We define the *synthetic analogue* for X to be the functor $\nu_X\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(X^{\tau})$ constructed in Proposition III.3.3.8. We will sometimes abusively just say the synthetic analogue and denote the functor by ν, if the choice of stack is clear from context.

Corollary III.3.3.10. *If X is a geometric spectral stack, then the synthetic analogue is unital. That is the map $\mathcal{O}_{X^{\tau}} \to \nu \mathcal{O}_X$ provided by the lax symmetric monoidal structure of ν is an equivalence.*

Proof. Let $p\colon \operatorname{Spec} R \to X$ be a flat cover, then the diagram

$$\begin{tikzcd}
\mathrm{QCoh}(X) \arrow[r, "\nu"] \arrow[d, "p^*"] & \mathrm{QCoh}(X^{\tau}) \arrow[d, "(p^{\tau})^*"] \\
\mathrm{Mod}_R(\mathcal{S}p) \arrow[r, "\tau_{\geqslant \bullet}"] & \mathrm{Mod}_{\tau_{\geqslant \bullet} R}(\mathcal{S}p^{\mathrm{fil}})
\end{tikzcd}$$

commutes by Proposition III.3.3.8, and the functor $(p^\tau)^*$ is conservative, so it suffices to see that $\tau_{\geqslant\bullet}\colon \mathrm{Mod}_R(\mathcal{S}p) \to \mathrm{Mod}_{\tau_{\geqslant\bullet}R}(\mathcal{S}p^{\mathrm{fil}})$ is unital, which it is by construction. $\square$

Proposition III.3.3.11. *If X is a geometric spectral stack, then the synthetic analogue $\nu\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(X^\tau)$ is fully faithful.*

Proof. Since fully faithful functors are closed under limits in Cat^{Δ^1} it suffices to see that $\tau_{\geqslant\bullet}\colon \mathrm{Mod}_R(\mathcal{S}p) \to \mathrm{Mod}_{\tau_{\geqslant\bullet}R}(\mathcal{S}p^{\mathrm{fil}})$ is fully faithful for every commutative ring spectrum R. To see this, recall that the Postnikov tower functor $\tau_{\geqslant\bullet}\colon \mathcal{S}p \to \mathcal{S}p^{\mathrm{fil}}$ can be defined as the composite of lax symmetric monoidal functors

$$\mathcal{S}p \xrightarrow{\mathrm{const}} \mathcal{S}p^{\mathrm{fil}} \xrightarrow{\tau^P_{\geqslant 0}} \mathcal{S}p^{\mathrm{fil}}_{\geqslant 0} \hookrightarrow \mathcal{S}p^{\mathrm{fil}}.$$

Hence, for any ring spectrum R, the functor

$$\tau_{\geqslant\bullet}\colon\ \mathrm{Mod}_R(\mathcal{S}p) \to \mathrm{Mod}_{\tau_{\geqslant\bullet}}(\mathcal{S}p^{\mathrm{fil}})$$

agrees with the composite

$$\mathrm{Mod}_R(\mathcal{S}p) \xrightarrow{\mathrm{const}} \mathrm{Mod}_{\tau_{\geqslant\bullet}R}(\mathcal{S}p^{\mathrm{fil}}) \xrightarrow{\tau^P_{\geqslant 0}} \mathrm{Mod}_{\tau_{\geqslant\bullet}R}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0}) \hookrightarrow \mathrm{Mod}_{\tau_{\geqslant\bullet}R}(\mathcal{S}p^{\mathrm{fil}}).$$

Since $\tau^P_{\geqslant 0}\colon \mathrm{Mod}_{\tau_{\geqslant\bullet}R}(\mathcal{S}p^{\mathrm{fil}}) \to \mathrm{Mod}_{\tau_{\geqslant\bullet}R}(\mathcal{S}p^{\mathrm{fil}})$ is right adjoint and we know that $\tau^{-1}\tau_{\geqslant\bullet} \simeq \mathrm{id}_{\mathcal{S}p}$, we see that

$$\begin{aligned}
\mathrm{map}_{\mathrm{Mod}_{\tau_{\geqslant\bullet}R}(\mathcal{S}p^{\mathrm{fil}})}(\tau_{\geqslant\bullet}M, \tau_{\geqslant\bullet}N) &\simeq \mathrm{map}_{\mathrm{Mod}_{\tau_{\geqslant\bullet}R}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})}(\tau^P_{\geqslant 0}\,\mathrm{const}\, M, \tau^P_{\geqslant 0}\,\mathrm{const}\, N) \\
&\simeq \mathrm{map}_{\mathrm{Mod}_{\tau_{\geqslant\bullet}R}(\mathcal{S}p^{\mathrm{fil}})}(\tau^P_{\geqslant 0}\,\mathrm{const}\, M, \mathrm{const}\, N) \\
&\simeq \mathrm{map}_{\mathrm{Mod}_R(\mathcal{S}p)}(\tau^{-1}\tau^P_{\geqslant 0}\,\mathrm{const}\, M, N) \\
&\simeq \mathrm{map}_{\mathrm{Mod}_R(\mathcal{S}p)}(M, N).
\end{aligned}$$

Here we have used that τ^{-1} is left adjoint to const in the third equivalence. $\square$

Remark III.3.3.12. The synthetic analogue for X a priori depends on a choice of flat cover of X. We will now see that the functor is independent of the choice of flat cover of X.

Proposition III.3.3.13. *Let X be a geometric spectral stack. If $p\colon \operatorname{Spec} R \to X$ and $q\colon \operatorname{Spec} S \to X$ are flat covers of X, then the synthetic analogue ν_p defined via p and the synthetic analogue ν_q defined via q are naturally equivalent.*

Proof. Since p and q are flat and hence affine. It follows that the fiber product $\operatorname{Spec} R \times_X \operatorname{Spec} S \simeq \operatorname{Spec} A$ is affine. Let us define the projections $\operatorname{Spec} A \to \operatorname{Spec} R$ and $\operatorname{Spec} A \to \operatorname{Spec} S$ by f_R and f_S respectively. Let $\operatorname{Spec} R^\bullet$, $\operatorname{Spec} S^\bullet$ and $\operatorname{Spec} A^\bullet$ denote the Čech nerves of p, q and $pf_R \simeq qf_S$ respectively. Using the same argument as in the proof of Proposition III.3.3.8 we obtain a map of spans

$$\begin{tikzcd}
\operatorname{Mod}_{R^\bullet}(\mathcal{S}p) \arrow[r, "f_R^\bullet"] \arrow[d, "\tau^\sharp_{\geqslant\bullet}"] & \operatorname{Mod}_{A^\bullet}(\mathcal{S}p) \arrow[d, "\tau^\sharp_{\geqslant\bullet}"] & \operatorname{Mod}_{S^\bullet}(\mathcal{S}p) \arrow[l, "f_S^\bullet"'] \arrow[d, "\tau^\sharp_{\geqslant\bullet}"] \\
\operatorname{Mod}_{\tau_{\geqslant\bullet}R^\bullet}(\mathcal{S}p^{\mathrm{fil}}) \arrow[r, "\tau_{\geqslant\bullet}f_R^\bullet"] & \operatorname{Mod}_{\tau_{\geqslant\bullet}A^\bullet}(\mathcal{S}p^{\mathrm{fil}}) & \operatorname{Mod}_{\tau_{\geqslant\bullet}S^\bullet}(\mathcal{S}p^{\mathrm{fil}}). \arrow[l, "\tau_{\geqslant\bullet}f_S^\bullet"']
\end{tikzcd}$$

It suffices to show that the horizontal maps induce equivalences upon taking the limit. This is clear by 2-out-of-3 for equivalences of categories. □

Proposition III.3.3.14. *If $f\colon X \to Y$ is a map of geometric spectral stacks, then the diagram*

$$\begin{tikzcd}
\operatorname{QCoh}(Y) \arrow[r, "\nu_Y"] \arrow[d, "f^*"] & \operatorname{QCoh}(Y^\tau) \arrow[d, "(f^\tau)^*"] \\
\operatorname{QCoh}(X) \arrow[r, "\nu_X"] & \operatorname{QCoh}(X^\tau)
\end{tikzcd}$$

commutes.

Proof. Choose a flat cover $p\colon \operatorname{Spec} R \to Y$, such that $X \times_Y \operatorname{Spec} R \simeq \operatorname{Spec} S$ is affine and defines a flat cover $q\colon \operatorname{Spec} S \to X$. Using the same argument as in the proof of Proposition III.3.3.8 we obtain a square

$$\begin{tikzcd}
\mathrm{Mod}_{R^\bullet}(\mathcal{S}p) \arrow[r] \arrow[d] & \mathrm{Mod}_{\tau_{\geqslant \bullet} R^\bullet}(\mathcal{S}p^{\mathrm{fil}}) \arrow[d] \\
\mathrm{Mod}_{S^\bullet}(\mathcal{S}p) \arrow[r] & \mathrm{Mod}_{\tau_{\geqslant \bullet} S^\bullet}(\mathcal{S}p^{\mathrm{fil}}),
\end{tikzcd}$$

giving the square in question upon taking limits. □

Remark III.3.3.15. Using *lax quasi-coherent sheaves*[1] the results of this section can be made fully functorial in stacks, providing a lax natural transformation

$$\mathrm{Lax}\,\mathrm{QCoh}(-) \Rightarrow \mathrm{Lax}\,\mathrm{QCoh}((-)^\tau)$$

of functors $\mathrm{Stk} \to \mathrm{CAlg}(\mathrm{Cat})^{\mathrm{lax}}$. The author might pursue this further in the future.

Theorem III.3.3.16. If X is a geometric spectral stack, then the diagram

$$\begin{tikzcd}
 & \mathrm{QCoh}(X) \arrow[d, "\nu"] \arrow[dl, bend right, "\mathrm{id}"'] \arrow[dr, bend left, "\pi_*"] & \\
\mathrm{QCoh}(X) & \mathrm{QCoh}(X^\tau) \arrow[l, "\tau^{-1}"'] \arrow[r, "\mathrm{gr}_*"] & \mathrm{QCoh}(X^{d\heartsuit})
\end{tikzcd}$$

commutes.

[1]Lax quasi-coherent sheaves is the theory one obtains from applying the lax Kan extension of $\mathrm{Mod}_-\colon \mathrm{CAlg}(\mathcal{S}p) \to \mathrm{CAlg}(\mathrm{Cat})$ along $\mathrm{Spec}(-)$ instead of Kan extending as per usual.

Proof. The functors sending a stack X to $\mathrm{QCoh}(X)$, $\mathrm{QCoh}(X^\tau)$, $\mathrm{QCoh}(X^\tau)[\tau^{-1}]$ and $\mathrm{gr}_*\mathrm{QCoh}(X^\tau) \simeq \mathrm{QCoh}(X^{d\heartsuit})$ respectively, all preserve limits, so we may reduce to the case where X is affine. In this situation the conclusion follows from the results in Section III.2.1. $\square$

III.3.4 Coherent cohomology

In this section we study the coherent cohomology of a filtered stack. The cohomology $\Gamma(X, \mathcal{F})$ of a filtered stack X with coefficients in a quasi-coherent sheaf $\mathcal{F}$ is a filtered spectrum. In this section we will study this filtered spectrum and show that the associated spectral sequence interpolates between the coherent cohomology of $\mathrm{gr}_*\mathcal{F} \in \mathrm{QCoh}(X^\heartsuit)$ and the cohomology of $\tau^{-1}\mathcal{F}$. Furthermore, in the special case where $X = Y^\tau$ is the associated filtered stack to a geometric spectral stack Y, and $\mathcal{F} = \nu\mathcal{G}$ is in the image of the synthetic analogue, then we will see that $\Gamma(Y^\tau, \nu\mathcal{G})$ is the filtered spectrum whose associated spectral sequence is the descent spectral sequence for $\Gamma(Y, \mathcal{G})$.

Before we define cohomology it is relevant for us to be precise about where the filtration arises from. This extra structure comes from the fact that for any filtered stack X, the category of quasi-coherent sheaves on X is $\mathcal{S}p^{\mathrm{fil}}$-linear in Pr^L. This can be made more coherent using enriched category theory, but the following construction suffices for the purposes of this paper.

Construction III.3.4.1. Let $\mathcal{C} \in \mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L)$ be a $\mathcal{S}p^{\mathrm{fil}}$-linear presentable category and consider the functor

$$- \otimes -\colon \mathcal{S}p^{\mathrm{fil}} \otimes \mathcal{C} \to \mathcal{C},$$

provided as part of this structure. For any object $c \in \mathcal{C}$ this determines a left adjoint functor

$$- \otimes c\colon \mathcal{S}p^{\mathrm{fil}} \to \mathcal{C},$$

whose right adjoint we denote by $\underline{\mathrm{map}}_{\mathcal{C}}(c,-)\colon \mathcal{C} \to \mathcal{S}p^{\mathrm{fil}}$. We will denote the k-th filtered piece of $\underline{\mathrm{map}}_{\mathcal{C}}(c,-)$ by $\mathrm{map}^k_{\mathcal{C}}(c,-)$. For any object $d \in \mathcal{C}$ we say that $\underline{\mathrm{map}}_{\mathcal{C}}(c,d)$ is the *filtered mapping spectrum* from c to d.

Similarly, if $\mathcal{C}$ is $\mathcal{S}p^{\mathrm{gr}}$-linear, then for every $c \in \mathcal{C}$ we obtain a *graded mapping spectrum* functor

$$\underline{\mathrm{map}}^{\mathrm{gr}}_{\mathcal{C}}(c,-)\colon \mathcal{C} \to \mathcal{S}p^{\mathrm{gr}}.$$

Proposition III.3.4.2. *Let $\mathcal{C} \in \mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L)$ be a $\mathcal{S}p^{\mathrm{fil}}$-linear presentable category. If $c, d \in \mathcal{C}$ are objects of, then the filtered mapping spectrum $\underline{\mathrm{map}}_{\mathcal{C}}(c,d)$ satisfies the following properties.*

1. *The k-th filtered piece is given by*

$$\mathrm{map}^k_{\mathcal{C}}(c,d) \simeq \mathrm{map}_{\mathcal{C}}(\mathbb{S}^{0,k} \otimes c, d).$$

In fact, $\underline{\mathrm{map}}_{\mathcal{C}}(c,-)$ refines to a bifunctor.

2. *If $c \in \mathcal{C}$ is $\mathcal{S}p^{\mathrm{fil}}$-atomic*[1]*, then we have equivalences*

$$\tau^{-1}\underline{\mathrm{map}}_{\mathcal{C}}(c,d) \xrightarrow{\sim} \mathrm{map}_{\mathcal{C}[\tau^{-1}]}(\tau^{-1}c, \tau^{-1}d)$$

and

$$\mathrm{gr}_*\underline{\mathrm{map}}_{\mathcal{C}}(c,d) \xrightarrow{\sim} \underline{\mathrm{map}}_{\mathrm{gr}_*\mathcal{C}}(\mathrm{gr}_* c, \mathrm{gr}_* d).$$

3. *If $\mathcal{C}_-\colon I \to \mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L)$ is a diagram with limit $\mathcal{C} \simeq \lim_{i\in I}\mathcal{C}_i$ and canonical projections $p_i\colon \mathcal{C} \to \mathcal{C}_i$, then the canonical map*

$$\underline{\mathrm{map}}_{\mathcal{C}}(c,d) \xrightarrow{\sim} \lim_{i\in I}\underline{\mathrm{map}}_{\mathcal{C}_i}(p_i c, p_i d)$$

[1] I.e the functor $\mathcal{S}p^{\mathrm{fil}} \xrightarrow{-\otimes c} \mathcal{C}$ is an internal left adjoint in $\mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L)$. That is the functor $\underline{\mathrm{map}}_{\mathcal{C}}(c,-)\colon \mathcal{C} \to \mathcal{S}p^{\mathrm{fil}}$ preserves colimits and the map $X \otimes \underline{\mathrm{map}}_{\mathcal{C}}(c,d) \to \underline{\mathrm{map}}_{\mathcal{C}}(c, X \otimes d)$ is an equivalence for every $X \in \mathcal{S}p^{\mathrm{fil}}$ and $d \in \mathcal{C}$.

is an equivalence. The same result holds if we replace $\mathcal{S}p^{\mathrm{fil}}$ with $\mathcal{S}p^{\mathrm{gr}}$ and the filtered mapping spectrum with the graded mapping spectrum.

Proof. We first prove (1). Recall that the functor $\mathrm{ev}_k \colon \mathcal{S}p^{\mathrm{fil}} \to \mathcal{S}p$ is corepresented by $\mathbb{S}^{0,k}$ and that $\underline{\mathrm{map}}_{\mathcal{C}}(c,-)$ satisfies the following universal property:

$$\mathrm{map}_{\mathcal{C}}(X \otimes c, d) \simeq \mathrm{map}_{\mathcal{S}p^{\mathrm{fil}}}(X, \underline{\mathrm{map}}_{\mathcal{C}}(c,d)).$$

Putting this together we see that

$$\begin{aligned} \mathrm{map}^k_{\mathcal{C}}(c,d) &\simeq \mathrm{map}_{\mathcal{S}p^{\mathrm{fil}}}(\mathbb{S}^{0,k}, \underline{\mathrm{map}}_{\mathcal{C}}(c,d)) \\ &\simeq \mathrm{map}_{\mathcal{C}}(\mathbb{S}^{0,k} \otimes c, d). \end{aligned}$$

We now prove (2). If $c \in \mathcal{C}$ is $\mathcal{S}p^{\mathrm{fil}}$-atomic, then $\mathcal{S}p^{\mathrm{fil}} \xrightarrow{-\otimes c} \mathcal{C}$ is an internal left adjoint. For any commutative algebra $A \in \mathcal{S}p^{\mathrm{fil}}$ the functor $\mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L) \to \mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L)^{\Delta^1}$, sending a $\mathcal{S}p^{\mathrm{fil}}$-module $\mathcal{M}$ to $M \to \mathrm{Mod}_A(\mathcal{S}p^{\mathrm{fil}}) \otimes_{\mathcal{S}p^{\mathrm{fil}}} \mathcal{M}$ refines canonically to a 2-functor. In particular sends internal left adjoints to internal left adjoints. As a consequence, we see that the square

$$\begin{tikzcd} \mathcal{S}p^{\mathrm{fil}} \arrow[r, "f^*"] \arrow[d, "-\otimes c"] & \mathrm{Mod}_A(\mathcal{S}p^{\mathrm{fil}}) \arrow[d, "-\otimes f^*c"] \\ \mathcal{C} \arrow[r, "\mathrm{Mod}_A(\mathcal{S}p^{\mathrm{fil}}) \otimes f^*"] & \mathrm{Mod}_A(\mathcal{S}p^{\mathrm{fil}}) \otimes_{\mathcal{S}p^{\mathrm{fil}}} \mathcal{C} \end{tikzcd}$$

is an internal left adjoint in $\mathrm{Mod}_{\mathcal{S}p^{\mathrm{fil}}}(\mathrm{Pr}^L)^{\Delta^1}$. This is equivalent to the above diagram being right adjointable by [AGH25, Corollary 5.2.12.]. If we set $A = \tau^{-1}\mathbb{S}^{0,0}$ or $A = C\tau$, then this proves the claim.

We now prove (3). We recall without proof that for every $c, d \in \mathcal{C}$, there is an equivalence

$$\mathrm{map}_{\mathcal{C}}(c,d) \xrightarrow{\sim} \lim_{i \in I} \mathrm{map}_{\mathcal{C}_i}(p_i c, p_i d) \tag{III.3.4.3}$$

of mapping spectra. Since the functors $\{\mathrm{ev}_k \colon \mathcal{S}p^{\mathrm{fil}} \to \mathcal{S}p\}_{k\in\mathbb{Z}}$ are jointly conservative and preserve limits, it suffices to construct a map

$$\underline{\mathrm{map}}_{\mathcal{C}}(c,d) \to \lim_{i\in I} \underline{\mathrm{map}}_{\mathcal{C}_i}(p_i c, p_i d),$$

which is equivalent to $\mathrm{map}_{\mathcal{C}}(\mathbb{S}^{0,k}\otimes c, d) \to \lim_{i\in I} \mathrm{map}_{\mathcal{C}_i}(\mathbb{S}^{0,k}\otimes c, d)$ after evaluation at $k\in\mathbb{Z}$.

To do so we consider the commutative diagram

$$\begin{tikzcd} & \mathcal{S}p^{\mathrm{fil}} \arrow[dl, "-\otimes c"'] \arrow[dr, "-\otimes c_i"] & \\ \mathcal{C} \arrow[rr, "p_i"] & & \mathcal{C}_i, \end{tikzcd}$$

induced by the universal cone provided by the identification $\mathcal{C} \xrightarrow{\sim} \lim_{i\in I}\mathcal{C}_i$. This induces a $\mathcal{S}p^{\mathrm{fil}}$-linear functor $f \colon \mathcal{S}p^{\mathrm{fil}} \to \lim_{i\in I}\mathcal{C}_i$, whose right adjoint is given by $f^R(\{M_i\}_{i\in I}) = \lim_{i\in I} f_i^R M_i$. Here f_i^R is the right adjoint of the composite

$$\mathcal{S}p^{\mathrm{fil}} \xrightarrow{-\otimes c} \mathcal{C} \xrightarrow{p_i} \mathcal{C}_i.$$

Note that by definition $f_i^R(-) \simeq \underline{\mathrm{map}}_{\mathcal{C}_i}(p_i c, -)$. If we apply the Yoneda lemma to the equivalence

$$\begin{aligned} \mathrm{map}_{\mathcal{S}p^{\mathrm{fil}}}(X, f^R(\{M_i\}_{i\in I})) &\simeq \mathrm{map}_{\mathcal{S}p^{\mathrm{fil}}}(X, \lim_{i\in I} f_i^R M_i) \\ &\simeq \lim_{i\in I} \mathrm{map}_{\mathcal{S}p^{\mathrm{fil}}}(X, \underline{\mathrm{map}}_{\mathcal{C}_i}(p_i c, M_i)) \\ &\simeq \lim_{i\in I} \mathrm{map}_{\mathcal{C}_i}(X\otimes p_i c, M_i) \\ &\simeq \lim_{i\in I} \mathrm{map}_{\mathcal{C}_i}(p_i(X\otimes c), M_i) \\ &\simeq \mathrm{map}_{\mathcal{C}}(X\otimes c, \lim_{i\in I} p_i^R M_i) \\ &\simeq \mathrm{map}_{\mathcal{S}p^{\mathrm{fil}}}(X, \underline{\mathrm{map}}_{\mathcal{C}}(c, \lim_{i\in I} p_i^R M_i)). \end{aligned}$$

Here the first equivalence follows from our identification of f^R, the second follows from the definition of $\underline{\mathrm{map}}_{\mathcal{C}_i}(p_i c, -)$, the third follows from the universal property of $\underline{\mathrm{map}}_{\mathcal{C}_i}(p_i c, -)$, the fourth follows from our assumption that p_i is $\mathcal{S}p^{\mathrm{fil}}$-linear for every $i \in I$, the fifth equivalence follows from Eq. (III.3.4.3) and the sixth equivalence is a consequence of the universal property of $\underline{\mathrm{map}}_{\mathcal{C}}(c, -)$.

It is clear that this proof also works for $\mathcal{S}p^{\mathrm{gr}}$, but it can also be reduced to this setting, by using the identification $\mathrm{Mod}_{C\tau}(\mathcal{S}p^{\mathrm{fil}}) \xrightarrow{\sim} \mathcal{S}p^{\mathrm{gr}}$ and the formula for mapping spaces in the category of module in Lemma A.1.10. □

Definition III.3.4.4. If X is a filtered stack and $\mathcal{F} \in \mathrm{QCoh}(X)$ is a quasi-coherent sheaf on X, then the *coherent cohomology* of X with coefficient in $\mathcal{F}$ is the filtered spectrum

$$\Gamma(X, \mathcal{F}) \coloneqq \underline{\mathrm{map}}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, \mathcal{F}).$$

Moreover, we define the coherent cohomology functor to be

$$\Gamma(X, -) \coloneqq \underline{\mathrm{map}}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, -)\colon \mathrm{QCoh}(X) \to \mathcal{S}p^{\mathrm{fil}}.$$

If X is a geometric spectral stack, we will use the notation

$$\Gamma^\tau(X, \mathcal{F}) \coloneqq \Gamma(X^\tau, \nu\mathcal{F}),$$

and say that $\Gamma^\tau(X, \mathcal{F})$ is the *filtered coherent cohomology* of X with coefficients in $\mathcal{F}$.

Remark III.3.4.5. If X is a filtered stack and $p\colon X \to \operatorname{Spec} \mathbb{S}^{0,0}$ is the map to the terminal object in FStack, then we have an identification

$$\Gamma(X, -) \simeq p_*(-).$$

This can be seen as follows. Any $\mathcal{S}p^{\mathrm{fil}}$-linear functor $f\colon \mathcal{S}p^{\mathrm{fil}} \to \mathcal{C}$ is determined by the image of $\mathbb{S}^{0,0}$, that is, there is an identification

$$\mathrm{Fun}^L_{\mathcal{S}p^{\mathrm{fil}}}(\mathcal{S}p^{\mathrm{fil}}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C},$$

induced by restriction along $\{\mathbb{S}^{0,0}\} \to \mathcal{S}p^{\mathrm{fil}}$. By construction $p^*\colon \mathcal{S}p^{\mathrm{fil}} \to \mathrm{QCoh}(X)$ is $\mathcal{S}p^{\mathrm{fil}}$-linear and symmetric monoidal, so $p^*(\mathbb{S}^{0,0}) \simeq \mathcal{O}_X$. Hence, by uniqueness of adjoints, we obtain an equivalence $\Gamma(X,-) \simeq p_*$.

One immediate consequence of this, is that $\Gamma(X,-)$ admits a canonical choice of lax symmetric monoidal structure. In particular, $\Gamma(X) \coloneqq \Gamma(X, \mathcal{O}_X)$ has a canonical commutative algebra structure in filtered spectra.

Corollary III.3.4.6. *If X is a filtered stack, with a presentation $\mathrm{colim}_{i\in I} \operatorname{Spec} R_i \xrightarrow{\sim} X$, then we have an equivalence of filtered spectra*

$$\Gamma(X,-) \simeq \lim_{i\in I^{\mathrm{op}}} \underline{\mathrm{map}}_{\mathrm{Mod}_{R_i}(\mathcal{S}p^{\mathrm{fil}})}(R_i, f_i^*(-)).$$

Here f_i is the canonical map $\operatorname{Spec} R_i \to X$.

Proof. Our assumption tells us that $\mathrm{QCoh}(X) \simeq \lim_{i\in I^{\mathrm{op}}} \mathrm{Mod}_{R_i}(\mathcal{S}p^{\mathrm{fil}})$. The conclusion follows from Proposition III.3.4.2. □

Remark III.3.4.7. If X is a filtered stack, then $\mathcal{O}_X$ is rarely $\mathcal{S}p^{\mathrm{fil}}$-atomic. If $\mathcal{O}_X$ is $\mathcal{S}p^{\mathrm{fil}}$-atomic, then it is compact in $\mathrm{QCoh}(X)$. This follows from the identification $\mathrm{map}^0_{\mathrm{QCoh}(X)}(\mathcal{O}_X,-) \simeq \mathrm{map}_{\mathrm{QCoh}(X)}(\mathcal{O}_X,-)$. In particular, the identification in Proposition III.3.4.2 part (2) does not hold in this case. If $X = Y^\tau$ is the filtered stack associated to a geometric spectral stack Y, then we will see in Proposition III.3.4.11 that the identification still holds true for quasi-coherent sheaves $\nu\mathcal{F}$ in the image of the synthetic analogue $\nu\colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(Y^\tau)$.

Definition III.3.4.8. Let X be a geometric spectral stack, let $p\colon \operatorname{Spec} R \to X$ be a flat cover of X and let $\operatorname{Spec} R^\bullet$ be the Čech nerve of p. If $\mathcal{F}$ is a quasi-coherent sheaf on X and $p_n\colon \operatorname{Spec} R^n \to X$ denotes the canonical projection, then the *descent spectral sequence* of $\mathcal{F}$ is the spectral sequence associated to the filtered spectrum:

$$\mathrm{DF}(X, \mathcal{F}) \coloneqq \lim_{[n]\in\Delta} \tau_{\geqslant\bullet} p_n^* \mathcal{F}.$$

Proposition III.3.4.9. *Let X be a geometric spectral stack, let $p\colon \operatorname{Spec} R \to X$ a flat cover and let $\mathcal{F}$ be a quasi-coherent sheaf on X. The descent spectral sequence of $\mathcal{F}$ satisfies the following properties.*

1. *The filtered spectrum $\mathrm{DF}(X, \mathcal{F})$ is complete and the underlying spectrum is equivalent to $\Gamma(X, \mathcal{F})$ the coherent cohomology of $\mathcal{F}$.*

2. *If $q\colon \operatorname{Spec} S \to X$ is a flat cover of X and we let $DF_p(X, \mathcal{F})$ be the filtered spectrum defined as in Definition III.3.4.8 with p and $\mathrm{DF}_q(X, \mathcal{F})$ be defined using q, then there is an equivalence of filtered spectra*
$$\mathrm{DF}_p(X, \mathcal{F}) \simeq \mathrm{DF}_q(X, \mathcal{F}),$$
and hence of associated spectral sequences.

3. *The E_1-page of the descent spectral sequence of $\mathcal{F}$ is given by*
$$E_1^{p,q} = \pi_{p,q}(\lim_{[n]\in\Delta} \Sigma^* \pi_*(p_n^* \mathcal{F})).$$

Proof. The completeness statement in (1) follows from the fact that $\tau_{\geqslant\bullet} X$ is complete for all spectra X and complete spectra are closed under limits in filtered spectra. The identification of the underlying spectrum of $DF(X, \mathcal{F})$ is done completely analogously to [CDN25, Prop. 1.6]. The statement in (2) can be proven completely analogously to [CDN25, Lemma 2.3]. The statement in (3) follows immediately from the fact that $\mathrm{gr}_*\colon \mathcal{S}p^{\mathrm{fil}} \to \mathcal{S}p^{\mathrm{gr}}$ preserves limits. □

Remark III.3.4.10. One might expect the E_1-page of the descent spectral sequence to be given by coherent cohomology for some classical (Dirac) stack, and this is the case. However, we will need the full deformation theoretic picture in order to make this identification coherently.

Proposition III.3.4.11. *Let X be a geometric spectral stack. If $\mathcal{F} \in \mathrm{QCoh}(X)$ is a quasi-coherent sheaf on X, then filtered coherent cohomology of X with coefficient in $\mathcal{F}$ is the filtered spectrum representing the descent spectral sequence for the pair $(X, \mathcal{F})$. That is, there is an equivalence of filtered spectra*

$$\Gamma^\tau(X, \mathcal{F}) = \mathrm{DF}(X, \mathcal{F}).$$

Proof. Let $p\colon \operatorname{Spec} R \to X$ be a flat cover of X and let $\operatorname{Spec} R^\bullet$ denote the Čech nerve of p. It follows from Corollary III.3.4.6 that we have an identification

$$\Gamma^\tau(X, \mathcal{F}) \xrightarrow{\sim} \lim_{[n]\in\Delta} \underline{\mathrm{map}}_{\mathrm{Mod}_{\tau_{\geqslant\bullet} R^n}(\mathcal{S}p^{\mathrm{fil}})}(\tau_{\geqslant\bullet} R^n, (\tau_{\geqslant\bullet} p_n)^* \nu\mathcal{F}).$$

Here $p_n\colon \operatorname{Spec} R^n \to X$ denotes the canonical map. It is easy to verify that $\underline{\mathrm{map}}_{\mathrm{Mod}_{\tau_{\geqslant\bullet} R^n}(\mathcal{S}p^{\mathrm{fil}})}(\tau_{\geqslant\bullet} R^n, -)$ can be identified with the forgetful functor

$$\mathrm{Mod}_{\tau_{\geqslant\bullet} R^n}(\mathcal{S}p^{\mathrm{fil}}) \to \mathcal{S}p^{\mathrm{fil}}.$$

Omitting the forgetful functor from the notation and using the naturality of the synthetic analogue shown in Proposition III.3.3.14, we see that

$$\Gamma^\tau(X, \mathcal{F}) \simeq \lim_{[n]\in\Delta} (\tau_{\geqslant\bullet} p_n)^* \nu\mathcal{F} \simeq \lim_{[n]\in\Delta} \tau_{\geqslant\bullet}(p_n^* \mathcal{F}) \simeq \mathrm{DF}(X, \mathcal{F}). \qquad \square$$

Remark III.3.4.12. We consider the above proposition as analogous to the following result. If E is an Adams-type ring spectrum and

$\nu\colon \mathcal{S}p \to \mathrm{Syn}_E$ is the synthetic analogue, then for any diagram $F\colon I \to \mathcal{S}p$, there is an identification $\tau^{-1}\lim_{i\in I}\nu F(i) \simeq \lim_{i\in I} F(i)$ of spectra. For a proof see [CDN25, Prop. 1.6].

Chapter III.4

Applications

In the previous two parts we have set up the necessary foundations for a well-functioning theory of filtered stacks. Using this machinery we will now put our focus on two examples. First, the theory of synthetic spectra as introduced by Pstrągowski in [Pst22], where we associate to any *Adams*-type commutative ring spectrum $E \in \mathrm{CAlg}(\mathcal{S}p)$ a geometric filtered stack $\mathcal{M}_E^\tau$ whose category of quasi-coherent sheaves provides a "geometric" analogue of E-based synthetic spectra. We will also provide a comparison functor

$$\mathrm{Syn}_E \to \mathrm{QCoh}(\mathcal{M}_E^\tau)$$

and study its properties.

We will then focus our attention on the even filtration introduced by Hahn–Raksit–Wilson in [HRW25]. The even filtration on a commutative ring spectrum R can be considered the descent spectral sequence associated to a certain stack on even commutative ring spectra. In Section III.4.4 we will show that this has a natural refinement as a filtered stack and study its properties. Finally, we will show that any spectral stack has a even filtered stack associated to it; this is a variation of the associated filtered stack functor $(-)^\tau \colon \mathrm{Stk} \to \mathrm{FStack}$ constructed in Theorem III.3.1.9.

III.4.1 Descent object of Adams type ring spectra

For a ring spectrum R and a spectrum X, the R-based Adams spectral sequence of X is the spectral sequence associated to the filtered spectrum

$$\mathrm{AF}_R(X) \coloneqq \lim_{[n]\Delta} \tau_{\geqslant \bullet} R^{\otimes n+1} \otimes X.$$

If R is a commutative ring spectrum, this is naturally seen to be the descent spectral sequence for a spectral stack $\mathcal{M}_R$. Furthermore, if $R \to R \otimes R$ is a π_*-flat map of ring spectra, then this stack is geometric. This stack can be seen to be (-1)-truncated in the category of spectral stacks. In particular, we obtain a fully faithful functor $\mathrm{Stk}_{/\mathcal{M}_R} \hookrightarrow \mathrm{Stk}$. This can be generalized to a relative story in the following way.

Definition III.4.1.1 ([BDL25, Definition 2.3.1.1])**.** Let $(\mathcal{C}, \tau, Q)$ be a quasi-coherent sheaf context. The *descent stack* $\mathcal{M}_f$ of a map $f \colon X \to Y$ is its (-1)-truncation in the category $\mathrm{Stk}(\mathcal{C})_{/Y}$. If Y is terminal in $\mathrm{Stk}(\mathcal{C})$, then we will write $\mathcal{M}_X$ for the descent stack of $f \colon X \to Y \simeq \mathrm{pt}$ instead. Furthermore, if $X \simeq \operatorname{Spec} R$ is affine, then we write $\mathcal{M}_R$.

Remark III.4.1.2. Our notation differs from the notation used by Balderrama–Davies–Linskens in [BDL25], where they denote the descent stack by D_f. We write $\mathcal{M}_f$, since we like to imagine this as some sort of moduli space. In the examples we care about, we will see that this is reasonable.

Remark III.4.1.3. For a map of stacks $f \colon X \to Y$, there is a factorization

$$X \xrightarrow{\hat{f}} \mathcal{M}_f \xrightarrow{i} Y,$$

where $\hat{f}$ is an effective epimorphism and i is a monomorphism.

Lemma III.4.1.4 ([BDL25, Lemma 2.3.1.2]). *For a map of stacks $f\colon X \to Y$ the following hold:*

1. *The descent stack of f is computed as the colimit of the Čech nerve of f. That is*

$$\operatorname{colim}_{k\in\Delta^{\mathrm{op}}} X^{\times_Y k+1} \xrightarrow{\sim} \mathcal{M}_f$$

is an equivalence.

2. *The forgetful functor $\mathrm{Stk}(\mathcal{C})_{/\mathcal{M}_f} \to \mathrm{Stk}(\mathcal{C})_{/Y}$ is fully faithful and preserves colimits and all non-empty limits.*

3. *If $Z, Z' \in \mathrm{Stk}(\mathcal{C})_{/\mathcal{M}_f}$, then the canonical map*

$$Z \times_{\mathcal{M}_f} Z' \to Z \times_Y Z'$$

is an equivalence.

4. *For a map $p\colon Z \to Y$ in $\mathrm{Stk}_{/Y}$, the following conditions are equivalent.*

   (a) *The map $p\colon Z \to Y$ is in the essential image of $\mathrm{Stk}(\mathcal{C})_{/\mathcal{M}_f} \to \mathrm{Stk}(\mathcal{C})_{/\mathcal{C}}$.*

   (b) *The map $Z \times_Y X \to Z$ is an effective epimorphism.*

   (c) *There exists an effective epimorphism $W \to Z$ and a commutative diagram*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r, "p"] & Y. \end{tikzcd}$$

Proposition III.4.1.5. *Let $f\colon X \to Y$ be a map of spectral stacks. If $X \times_Y X \to X$ is flat and X is a geometric stack, then $\mathcal{M}_f$ is a geometric stack.*

Proof. We fix a flat cover $p\colon \operatorname{Spec} R \to X$. By construction $X \to \mathcal{M}_f$ is an effective epimorphism, so the composite $\operatorname{Spec} R \to X \to \mathcal{M}_f$ is also an effective epimorphism. Therefore, it suffices to see that $X \to \mathcal{M}_f$ is flat. Being flat is a local property with respect to effective epimorphisms by Lemma III.3.1.14 so it suffices to see that $X \times_{\mathcal{M}_f} X \to X$ is flat. By Lemma III.4.1.4 (3) the horizontal map in the diagram

$$\begin{tikzcd}
X \times_{\mathcal{M}_f} X \arrow[rr, "\sim"] \arrow[dr] & & X \times_Y X \arrow[dl] \\
& X &
\end{tikzcd}$$

is an equivalence. In particular, $X \times_{\mathcal{M}_f} X \to X$ is flat if and only if $X \times_Y X \to X$ is. $\square$

Corollary III.4.1.6. *If $E \in \mathrm{CAlg}(\mathcal{S}p)$ is a commutative ring spectrum and $E \to E \otimes E$ is π_*-flat, then the descent stack $\mathcal{M}_E$ is a geometric spectral stack. In particular, the associated filtered stack $\mathcal{M}_E^{\tau}$ is a geometric filtered stack.*

Remark III.4.1.7. The condition that $E \to E \otimes E$ is π_*-flat, is exactly the condition one classically asks for in order to identify the E_2-page of the Adams spectral sequence with coherent cohomology of the stack $\mathcal{M}_E^{\heartsuit}$ cf. [HP24, Theorem 5.5]. So it is no surprise, that this is exactly the condition that is needed for $\mathcal{M}_E$ to be geometric, and thus makes the machinery from Section III.3.3 and Section III.3.4 work.

The following statement is due to Balderrama–Davies–Linskens and we thank them for allowing to us include a proof in this paper.

Proposition III.4.1.8. *Let* MU *denote the complex bordism spectrum, and consider the descent stack* $\mathcal{M}_{\mathrm{MU}}$*. If* $A \in \mathrm{CAlg}(\mathcal{S}p)$ *is any commutative ring spectrum, then*

$$\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\,A, \mathcal{M}_{\mathrm{MU}}) \simeq \begin{cases} * & \textit{if } A \textit{ is complex orientable,} \\ \emptyset & \textit{otherwise.} \end{cases}$$

Proof. Since $\mathcal{M}_{\mathrm{MU}}$ is (-1)-truncated by definition, then it suffices to see that $\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\,A, \mathcal{M}_{\mathrm{MU}})$ is non-empty if and only if A is complex orientable.

By Lemma III.4.1.4 there exists a map $\mathrm{Spec}\,A \to \mathcal{M}_{\mathrm{MU}}$ if and only if there exists a flat cover $\mathrm{Spec}\,B \to \mathrm{Spec}\,A$, and a commutative square

$$\begin{tikzcd} \mathrm{Spec}\,B \arrow[r] \arrow[d] & \mathrm{Spec}\,\mathrm{MU} \arrow[d] \\ \mathrm{Spec}\,A \arrow[r] & \mathcal{M}_{\mathrm{MU}}. \end{tikzcd}$$

In particular, B is complex orientable and $A \to B$ is faithfully π_*-flat. Now, since $A \to B$ is faithfully π_*-flat, it is injective on homotopy groups, so [BDL25, Lemma 4.1.2.3] applies, and we see that A is complex orientable. □

Remark III.4.1.9. In the proof above we see that B admits an $\mathbb{E}_\infty$-complex orientation. However, we cannot assure that the same holds for A.

Remark III.4.1.10. The map $\mathcal{M}_{\mathrm{MU}} \to \mathrm{Spec}\,\mathbb{S}$ defines a symmetric monoidal left adjoint

$$\mathcal{S}p \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{MU}})$$

where the unit of the adjunction is exactly the map

$$X \to \lim_{[k]\in\Delta} \mathrm{MU}^{\otimes k+1} \otimes X.$$

This map is the MU-nilpotent completion of X. By Quillen's theorem, this is an equivalence on the sphere. In particular, if we let $C(\mathcal{M}_{\mathrm{MU}})$ denote the smallest stable subcategory of $\mathrm{QCoh}(\mathcal{M}_{\mathrm{MU}})$ containing $\mathcal{O}_{\mathcal{M}_{\mathrm{MU}}}$, then we obtain a symmetric monoidal equivalence

$$\mathcal{S}p \xrightarrow{\sim} \operatorname{Ind} C(\mathcal{M}_{\mathrm{MU}}),$$

by the Schwede–Schipley theorem. This result is analogous to work of Gregoric [Gre21, Thm. 2.4.4], who proves an analogous result, working in the flat topology, in the case of $M_{\mathrm{FG}}^{\mathrm{op}}$ the moduli stack of oriented formal groups, which can be seen to be the descent stack of the periodic complex bordism spectrum MP in this topology. For this reason, one might be tempted to dub the descent stack $\mathcal{M}_{\mathrm{MU}}$, the spectral moduli stack of formal groups. We refrain from doing so, since this is not the focus of this paper and there already exists several spectral stack with similar names in the literature.

Warning III.4.1.11. If E is a commutative ring spectrum, then $\mathcal{M}_E^\tau$ need not be equivalent to $\mathcal{M}_{\tau_{\geqslant \bullet} E}$. Essentially, this is because $\operatorname{Spec}^\tau \mathbb{S} \simeq \operatorname{Spec} \tau_{\geqslant \bullet}\mathbb{S} \neq \operatorname{Spec} \mathbb{S}^{0,0}$. It is for this reason we needed the additional flexibility of working with geometric filtered stacks, instead of just descent stacks.

Theorem III.4.1.12. If $E \in \mathrm{CAlg}(\mathcal{S}p)$ is an Adams-flat commutative ring spectrum, i.e $E \to E \otimes E$ is π_*-flat, then $\mathrm{QCoh}(\mathcal{M}_E^\tau)$ is a 1-parameter deformation of the E-based Adams spectral sequence, in the following sense: If $\mathcal{F}_E\colon \mathcal{S}p \to \mathrm{QCoh}(\mathcal{M}_E)$ denotes the functor induced by $\mathcal{M}_E \to \operatorname{Spec} \mathbb{S}$, sending a spectrum X to its E-descent data $\mathcal{F}_E X \coloneqq \{E^{\otimes k+1} \otimes X\}_{[k]\in\Delta}$, then the following hold:

1. The coherent cohomology of $\mathcal{F}_E X$ is given by the E-nilpotent completion of X. That is, there is an equivalence

$$\Gamma(\mathcal{M}_E, \mathcal{F}_E X) \simeq \lim_{[k]\in\Delta} E^{\otimes k+1} \otimes X.$$

2. The filtered coherent cohomology of $E^\vee X$ is given by the E-based Adams spectral sequence. More precisely, there is an equivalence

$$\Gamma^\tau(\mathcal{M}_E, \mathcal{F}_E X) = \Gamma(\mathcal{M}_E^\tau, \nu\mathcal{F}_E X) \simeq \lim_{[k]\in\Delta} \tau_{\geqslant\bullet}(E^{\otimes k+1} \otimes X).$$

3. The associated graded of the filtered coherent cohomology of $E^\vee X$ is given by the coherent cohomology of the quasi-coherent sheaf

$$\mathcal{F}_{E_*}X \coloneqq \{\pi_*(E^{\otimes k+1} \otimes X)\}_{[k]\in\Delta}$$

on $X^\heartsuit$. That is there is an equivalence

$$\mathrm{gr}_*\Gamma^\tau(\mathcal{M}_E, \mathcal{F}_E X) \simeq \Gamma(X^\heartsuit, \mathcal{F}_{E_*}X).$$

Proof. To see (1), note that $\mathrm{QCoh}(\mathcal{M}_E) \simeq \lim_{[n]\in\Delta} \mathrm{Mod}_{E^{\otimes n+1}}(\mathcal{S}p)$ and by definition coherent cohomology is given by the right adjoint to $\mathcal{S}p \to \mathrm{QCoh}(\mathcal{M}_E)$, using the formula of the right adjoint from [HY17, Theorem 5.5]. We prove (2) using the same ideas, the only additional thing needed is the naturality of the synthetic analogue proven in Proposition III.3.3.14. The associated graded of the filtered coherent cohomology of $\mathcal{F}_E X$ can be identified via direct inspection, since $\mathrm{gr}_*\colon \mathcal{S}p^{\mathrm{fil}} \to \mathcal{S}p^{\mathrm{gr}}$ preserves limits, and the fact that $\mathrm{Spec}\,\pi_* E \to X^\heartsuit$ can be seen to be a flat cover. $\square$

III.4.2 Comparison with synthetic spectra

For this section we fix an Adams-type commutative ring spectrum $E \in \mathrm{CAlg}(\mathcal{S}p)$ and study the category of E-based synthetic spectra Syn_E introduced by Pstrągowski in [Pst22]. In order to do so we construct a lax symmetric monoidal comparison functor $\mathrm{Syn}_E \to \mathrm{QCoh}(\mathcal{M}_E^\tau)$, which we will show to be fully faithful on a certain subcategory of Syn_E. We will also see that the obstruction for this functor to be fully faithful has to do with convergence of the E-based Adams spectral sequence. First, we recall the construction of the category of E-based synthetic spectra and some basic properties.

Definition III.4.2.1 (Pstrągowski)**.** We say that a finite spectrum Y is *E-finite projective* if its E-homology E_*Y is finitely generated and projective as an E_*-module. We denote the subcategory spanned by E-finite projective spectra by $\mathcal{S}p_E^{\mathrm{fp}}$. An *$E$-based synthetic spectrum* is an additive functor $X\colon (\mathcal{S}p_E^{\mathrm{fp}})^{\mathrm{op}} \to \mathcal{S}p$, such that if $A \to B \to C$ is an exact sequence of E-finite projective spectra where the map $E_*B \to E_*C$ is surjective, then the sequence

$$X(C) \to X(B) \to X(A)$$

is an exact sequence of spectra. We denote the category of E-based synthetic spectra by Syn_E.

Remark III.4.2.2. The above definition is equivalent to the definition given by Pstrągowski in [Pst22], but the definition in loc.cit. is a little bit more refined. Pstrągowski shows that the collection of maps of E-finite projective spectra, which are surjective on E-homology, define a Grothendieck topology on $\mathcal{S}p_E^{\mathrm{fp}}$ in which covers are the singletons given by maps which are surjective on E-homology. Pstrągowski then defines the category of E-based synthetic spectra to be the category of additive $\mathcal{S}p$-valued sheaves on $\mathcal{S}p_E^{\mathrm{fp}}$. The definition above is seen to be equivalent to the sheaf condition above for additive sheaves [Pst22, Theorem 2.8]. While equivalent to the definition above, the definition as a sheaf category immediately provides several excellent categorical properties.

Remark III.4.2.3. The category of E-based synthetic spectra can be defined even when E is only a homotopy associative ring spectrum, but this is not the case for the category $\mathrm{QCoh}(\mathcal{M}_E^{\tau})$. For this reason, we have assumed that E is a commutative ring spectrum.

Proposition III.4.2.4 ([Pst22, Section 4])**.** *The category of E-based synthetic spectra satisfies the following properties.*

1. *The category* Syn_E *is a presentable stable category, which admits a symmetric monoidal structure induced via Day-convolution, making* Syn_E *into a presentably symmetric monoidal stable category.*

2. *There exists a lax symmetric monoidal functor* $\nu\colon \mathcal{S}p \to \mathrm{Syn}_E$*, which is symmetric monoidal on E-finite projective spectra. This functor is named the synthetic analogue. Furthermore, the synthetic analogue preserves filtered colimits. The synthetic analogue does not preserve colimits in general; in particular, we obtain bigraded spheres* $\mathbb{S}^{a,b} = \Sigma^a \nu(\mathbb{S}^b)$[1]*. There exists a map* $\tau\colon \mathbb{S}^{0,-1} \to \mathbb{S}^{0,0}$*, which can be shown to make* Syn_E *into a commutative algebra under* $\mathcal{S}p^{\mathrm{fil}}$ *in* Pr^L[2]*.*

3. *The category* Syn_E *is generated as a stable category by* νP*, where P is E-finite projective.*

4. *The category of E-based synthetic spectra admits a right complete t-structure, which is compatible with the symmetric monoidal structure and filtered colimits. Furthermore, there is an equivalence* $\mathrm{Syn}_E^{\heartsuit} \simeq \mathrm{Comod}_{E_*E}$ *between the heart of* Syn_E *and the category of E_*E-comodules.*

5. *The generic fiber of* Syn_E *is symmetric monoidally equivalent to the category of spectra and the special fiber is symmetric monoidally equivalent to Hovey's category of stable E_*E-comodules* Stable_{E_*E}*, whenever* Stable_{E_*E} *is generated under colimits by*

[1] This grading convention is different from the one used by Pstrągowski; we choose this one since it matches our choice in filtered spectra.

[2] This is proven in [BHS26, Appendix C].

E_*P, where P is E-finite projective. This happens for instance if E is Landweber exact.

Remark III.4.2.5. If Syn_E is generated by the bigraded spheres, then we say that it is *cellular*. In this situation it follows from the filtered Schwede-Shipley theorem [Pst23b, Prop. 3.16], that there is an equivalence

$$\mathrm{Mod}_{\sigma\nu\mathbb{S}}(\mathcal{S}p^{\mathrm{fil}}) \xrightarrow{\sim} \mathrm{Syn}_E\,.$$

Here $\sigma\colon \mathrm{Syn}_E \to \mathcal{S}p^{\mathrm{fil}}$ is right adjoint to the symmetric monoidal left adjoint provided in (2) above. This functor is named the *signature* in [CDN25, Section 1.4]. The category of E-based synthetic spectra is known to be cellular whenever E is connective as a spectrum see [Law24]. It is currently unknown if the category of E_n-based synthetic spectra is cellular. Here E_n is Morava E-theory of height n.

Remark III.4.2.6. If E is the complex bordism spectrum MU, then we have identifications

$$\sigma\nu_{\mathrm{MU}}\mathbb{S} = \lim_{\Delta} \tau_{\geqslant\bullet}\mathrm{MU}^{\otimes\bullet+1},$$

which is equivalent to $\Gamma(\mathcal{M}^{\tau}_{\mathrm{MU}})$ by Theorem III.4.1.12.

Theorem III.4.2.7. The E-based synthetic spectrum νE is a commutative algebra in Syn_E, since the synthetic analogue is lax symmetric monoidal. The category $\mathrm{Mod}_{\nu E}(\mathrm{Syn}_E)$ of νE-modules in synthetic spectra is generated under colimits by the objects $\Sigma^{a,b}\nu E$.

Proof. It follows from Proposition III.4.2.4 that $\mathrm{Mod}_{\nu E}(\mathrm{Syn}_E)$ is generated by the objects $\nu E \otimes \nu P$, where $P \in \mathcal{S}p_E^{\mathrm{fp}}$. Furthermore, it follows from [Pst22, Lemma 4.24] that $\nu E \otimes \nu P \simeq \nu(E \otimes P)$. Now, E_*P is finitely generated projective over E_* so there exists a surjection

$$\phi\colon \bigoplus_{i=0}^{n} E_*[k_i] \to E_*P$$

of E_*-modules. Here $[-]$ is the internal shift in E_*-modules. We claim that this lifts to a map $\tilde{\phi}\colon \bigoplus_{i=0}^{n} \Sigma^{k_i} E \to E \otimes P$, of E-module spectra. To see this, note that since $E \otimes P$ is an E-module, the E-based Adams spectral sequence degenerates at the E_1-page and is concentrated on the $t = 0$ row. In particular we obtain an equivalence

$$[\Sigma^{k_i} E, E \otimes P]_E \cong \mathrm{Hom}_{E_*}(E_*[k_i], E_*P).$$

Here we have used that $E_*(E \otimes P) \simeq E_*E \otimes_{E_*} E_*P$ is a cofree comodule. In particular, the map ϕ is surjective on homotopy by construction. Since E_*P is projective, ϕ has a splitting ψ, which we may lift to a map

$$\tilde{\psi}\colon E \otimes P \to \bigoplus_{i=0}^{n} \Sigma^{k_i} E$$

using the same argument. It follows from the naturality of the E-based Adams spectral sequence, that $\tilde{\phi}\tilde{\psi} = \mathrm{Id}_{E \otimes P}$. In particular, we see that $\nu E \otimes P$ is a retract of $\bigoplus_{i=0}^{n} \nu \Sigma^{k_i} E$. This proves the claim. $\square$

Corollary III.4.2.8. *There is a symmetric monoidal equivalence*

$$\mathrm{Mod}_{\tau_{\geqslant \bullet} E}(\mathcal{S}p^{\mathrm{fil}}) = \mathrm{Mod}_{\sigma \nu E}(\mathcal{S}p^{\mathrm{fil}}) \xrightarrow{\sim} \mathrm{Mod}_{\nu E}(\mathrm{Syn}_E).$$

Furthermore, this equivalence is t-exact with respect to the homological t-structure on Syn_E and the Postnikov t-structure on $\mathcal{S}p^{\mathrm{fil}}$.

Proof. The identification $\tau_{\geqslant \bullet} E \simeq \sigma \nu E$ is proven in [CDN25, Lemma 1.19] and the equivalence follows from the filtered Schwede-Shipley theorem [Pst23b, Prop. 3.16]. It remains to be seen that σ is symmetric monoidal. To see this note that its left adjoint is a symmetric monoidal equivalence, hence σ is symmetric monoidal.

We now show that the equivalence is t-exact. Recall that νE and $\tau_{\geqslant \bullet} E$ are connective in their respective categories and an object of $\mathrm{Mod}_{\nu E}(\mathrm{Syn}_E)$ (respectively $\mathrm{Mod}_{\tau_{\geqslant \bullet} E}(\mathcal{S}p^{\mathrm{fil}})$) is (co)connective if its image under the forgetfull functor is (co)connective in Syn_E (respectively $\mathcal{S}p^{\mathrm{fil}}$), by Proposition A.1.11. Furthermore, an object $X \in \mathrm{Syn}_E$ is connective if and only $\sigma(\nu E \otimes X)$ is connective in the Postnikov t-structure by [Pst22, Cor. 4.19]. This immediately shows that

$$\sigma\colon \mathrm{Mod}_{\nu E}(\mathrm{Syn}_E) \to \mathrm{Mod}_{\tau_{\geqslant \bullet} E}(\mathcal{S}p^{\mathrm{fil}})$$

is right t-exact, since if $X \in \mathrm{Mod}_{\nu E}(\mathrm{Syn}_E)$ is connective, then $\sigma(\nu E \otimes X)$ is connective, but σX is a retract of $\sigma(\nu E \otimes X)$ and thus connective. It remains to be shown that σ is left t-exact. In order to see this we will instead show that its left adjoint

$$\sigma^L\colon \mathrm{Mod}_{\tau_{\geqslant \bullet} E}(\mathcal{S}p^{\mathrm{fil}}) \to \mathrm{Mod}_{\nu E}(\mathrm{Syn}_E)$$

is right t-exact. By combining Proposition III.2.1.26 and Proposition III.2.1.23 we see that the Postnikov t-structure on $\mathrm{Mod}_{\tau_{\geqslant \bullet} R}(\mathcal{S}p^{\mathrm{fil}})$ is generated by $\{\Sigma^{n,n}\tau_{\geqslant \bullet} E\}_{n\in\mathbb{Z}}$, so since σ^L is colimit preserving and satisfies $\sigma^L(\Sigma^{n,n}\tau_{\geqslant \bullet} E) \simeq \Sigma^{n,n}\nu E$ for all $n \in \mathbb{Z}$, we see that it is right t-exact. $\square$

Remark III.4.2.9. The result in Theorem III.4.2.7 tell us that even though Syn_E need not be cellular, the category of modules over νE is. This is reminiscent of the story for $\mathrm{QCoh}(\mathcal{M}_E^\tau)$, which is rarely equivalent to $\mathrm{Mod}_{\Gamma(\mathcal{M}_E^\tau, \nu \mathcal{O}_{\mathcal{M}_E})}$. However, since $p^\tau\colon \mathrm{Spec}^\tau E \to \mathcal{M}_E^\tau$ is a flat cover (and thus affine), we see from Proposition III.3.2.6, that there is an equivalence

$$\mathrm{Mod}_{\tau_{\geqslant \bullet} E}(\mathcal{S}p^{\mathrm{fil}}) \xrightarrow{\sim} \mathrm{Mod}_{p_*^\tau \tau_{\geqslant \bullet} E}(\mathrm{QCoh}(\mathcal{M}_E^\tau)).$$

In particular, we see that Syn_E and $\mathrm{QCoh}(\mathcal{M}_E^\tau)$ are equivalent after base-change to νE and $p_*^\tau \tau_{\geqslant \bullet} E$ respectively. This observation will form the basis for constructing our comparison map.

In forthcoming work with Torgeir Aambø and Sven Van Nigtevecht we prove the following theorem.

Theorem III.4.2.10 (Aambø–N.–Van-Nigtevecht)**.** Let $f\colon E \to E'$ be a map of Adams-type ring spectra, then there is an inclusion of sites $\mathcal{S}p_E^{\mathrm{fp}} \subseteq \mathcal{S}p_{E'}^{\mathrm{fp}}$, which induces a functor

$$f_!\colon \mathrm{Syn}_E \to \mathrm{Syn}_{E'},$$

such that there is an equivalence $f_!\nu_E X \simeq \nu_{E'} X$ for every $X \in \mathcal{S}p$, which can be written as a filtered colimit of E-finite projective spectra or admits a homotopy E-module structure.

Theorem III.4.2.11. If $E \in \mathrm{CAlg}(\mathcal{S}p)$ is of Adams-type, then there is a colimit preserving symmetric monoidal functor $\mu\colon \mathrm{Syn}_E \to \mathrm{QCoh}(\mathcal{M}_E^\tau)$, such that the following statements hold.

1. The functor μ is t-exact.

2. The composite

$$\mathcal{S}p_E^{\mathrm{fp}} \xrightarrow{\nu} \mathrm{Syn}_E \xrightarrow{\mu} \mathrm{QCoh}(\mathcal{M}_E^\tau)$$

   is exact. That is, if $A \to B \to C$ is an exact sequence of E-finite projective spectra, where $E_*B \to E_*C$ is surjective, then the sequence

$$\mu\nu(A) \to \mu\nu(B) \to \mu\nu(C)$$

   is exact in $\mathrm{QCoh}(\mathcal{M}_E^\tau)$.

3. The composite

$$\mathcal{S}p \xrightarrow{\nu} \mathrm{Syn}_E \xrightarrow{\mu} \mathrm{QCoh}(\mathcal{M}_E^\tau) \xrightarrow{\Gamma(\mathcal{M}_E^\tau, -)} \mathcal{S}p^{\mathrm{fil}}$$

   sends a spectrum X to its Adams-filtered spectrum

$$\mathrm{AF}_E(X) \coloneqq \lim_{\Delta} \tau_{\geqslant \bullet}(E^{\otimes \bullet+1} \otimes X).$$

Furthermore, this is natural in the following sense. If $f\colon E \to E'$ is a map of Adams-type commutative ring spectra, then the square

$$\begin{tikzcd} \mathrm{Syn}_E \arrow[r, "f_!"] \arrow[d, "\mu_E"] & \mathrm{Syn}_{E'} \arrow[d, "\mu_{E'}"] \\ \mathrm{QCoh}(\mathcal{M}_E^\tau) \arrow[r, "(f^\tau)^*"] & \mathrm{QCoh}(\mathcal{M}_{E'}^\tau) \end{tikzcd}$$

commutes. Here we abuse notation and write $f\colon \mathcal{M}_{E'} \to \mathcal{M}_E$ for the map induced by $\mathrm{Spec}(f)\colon\ \mathrm{Spec}\, E' \to \mathrm{Spec}\, E$ via 0-truncation.

Proof. We first construct the functor μ. Consider the symmetric monoidal left adjoint $\rho\colon \mathcal{S}p^{\mathrm{fil}} \to \mathrm{Syn}_E$ defined in Proposition III.4.2.4 (2), if we apply Lemma III.3.3.7 to ρ, we obtain a natural transformation

$$\eta\colon\ \mathrm{Mod}_{-}(\mathcal{S}p^{\mathrm{fil}}) \to \mathrm{Mod}_{\rho(-)}(\mathrm{Syn}_E)$$

of functors $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}) \to \mathrm{CAlg}(\mathrm{Pr}^L)$. Note that $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})^{\rho^\flat} = \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})$ since ρ is colimit preserving and symmetric monoidal. It follows from Corollary III.4.2.8 that η_A is an equivalence for every $\tau_{\geqslant\bullet}E$-algebra A, since $\rho\colon\ \mathrm{Mod}_{\tau_{\geqslant\bullet}E}(\mathcal{S}p^{\mathrm{fil}}) \to \mathrm{Mod}_{\nu E}(\mathrm{Syn}_E)$ is a symmetric monoidal equivalence. In particular, if $\mathrm{Spec}\, E^\bullet$ denotes the Čech nerve of $\mathrm{Spec}\, E \to \mathrm{Spec}\,\mathbb{S}$, then there is a natural equivalence

$$\mathrm{Mod}_{\tau_{\geqslant\bullet}(E^\bullet)}(\mathcal{S}p^{\mathrm{fil}}) \to \mathrm{Mod}_{\nu E^\bullet}(\mathrm{Syn}_E)$$

of functors $\mathbb{\Delta} \to \mathrm{CAlg}(\mathrm{Pr}^L)$, so we obtain a symmetric monoidal equivalence

$$\mathrm{QCoh}(\mathcal{M}_E^\tau) \to \lim_{\mathbb{\Delta}} \mathrm{Mod}_{\nu E^\bullet}(\mathrm{Syn}_E).$$

We let μ denote the composite:

$$\mathrm{Syn}_E \to \lim_{\mathbb{\Delta}} \mathrm{Mod}_{\nu E^\bullet}(\mathrm{Syn}_E) \xrightarrow{\sim} \lim_{\mathbb{\Delta}} \mathrm{Mod}_{\tau_{\geqslant\bullet}(E^{\otimes \bullet+1})}(\mathcal{S}p^{\mathrm{fil}}) = \mathrm{QCoh}(\mathcal{M}_E^\tau).$$

It is clear from our construction that μ is colimit preserving and symmetric monoidal.

We now prove (1). We first prove that μ is right t-exact. Suppose X is a connective synthetic spectrum. In this situation it follows from [Pst22, Cor. 4.19] that $\sigma(\nu E \otimes X)$ is connective. It follows from the way we constructed μ that if $p\colon \operatorname{Spec} E \to \mathcal{M}_E$, then we have an equivalence $p^\tau_* \mu(X) = \sigma(\nu E \otimes X)$. In particular, X is connective, since p^τ_* creates connectivity by Theorem III.3.2.12. Suppose now that X is coconnective, and let $\mathcal{F}$ be a connective quasi-coherent sheaf on $\mathrm{QCoh}(\mathcal{M}^\tau_E)$. In this situation, we have equivalences:

$$\begin{aligned}
\mathrm{Map}_{\mathrm{QCoh}(\mathcal{M}^\tau_E)}(\mathcal{F}, \Omega\mu X) &\simeq \lim_{[n]\in\Delta} \mathrm{Map}_{\tau_{\geqslant\bullet}(E^{\otimes n+1})}((p^\tau_n)^*\mathcal{F}, \Omega(p^\tau_n)^*\mu X) \\
&\simeq \lim_{[n]\in\Delta} \mathrm{Map}_{\tau_{\geqslant\bullet}(E^{\otimes n+1})}((p^\tau_n)^*\mathcal{F}, \Omega\sigma(\nu E^{\otimes n+1} \otimes X)) \\
&\simeq \lim_{[n]\in\Delta} \mathrm{Map}_{\nu E^{\otimes n+1}}(\rho((p^\tau_n)^*\mathcal{F}), \Omega\nu E^{\otimes n+1} \otimes X).
\end{aligned}$$

We proved in Corollary III.4.2.8 that $\sigma\colon \mathrm{Mod}_{\nu E}(\mathrm{Syn}_E) \xrightarrow{\sim} \mathrm{Mod}_{\tau_{\geqslant\bullet}E}(\mathcal{S}p^{\mathrm{fil}})$ is t-exact. In particular, $\rho((p^\tau_n)^*\mathcal{F})$ is connective for every n, so we see that $\mathrm{Map}_{\nu E^{\otimes n+1}}(\rho((p^\tau_n)^*\mathcal{F}), \Omega\nu E^{\otimes n+1} \otimes X) \simeq 0$, since $\nu E^{\otimes n+1} \otimes X$ is coconnective by applying Corollary III.4.2.8 inductively.

We now prove (2). By the above, μ preserves colimits so it is exact. Furthermore, ν is exact by [Pst22, Lemma 4.23].

We now prove (3). For any spectrum X we know from [Pst22, Lemma 4.24] that $\nu(E \otimes X) \simeq \nu E \otimes \nu X$, since E is Adams-type. Furthermore, it follows from [CDN25, Lemma 1.19] that $\sigma(\nu(E \otimes X)) \simeq \tau_{\geqslant\bullet}(E \otimes X)$. Putting these facts together with Corollary III.3.4.6 we see that there are equivalences:

$$\begin{aligned}
\Gamma(\mathcal{M}^{\tau}_{E}, \mu\nu X) &\simeq \lim_{[n]\in\Delta} \underline{\mathrm{map}}_{\tau_{\geqslant\bullet}(E^{\otimes n+1})}(\tau_{\geqslant\bullet}(E^{\otimes n+1}), (p^{\tau}_{n})^{*}\mu\nu X) \\
&\simeq \lim_{[n]\in\Delta} \underline{\mathrm{map}}_{\tau_{\geqslant\bullet}(E^{\otimes n+1})}(\tau_{\geqslant\bullet}(E^{\otimes n+1}), \sigma(\nu E^{\otimes n+1} \otimes \nu X)) \\
&\simeq \lim_{[n]\in\Delta} \sigma(\nu E^{\otimes n+1} \otimes \nu X) \\
&\simeq \lim_{[n]\in\Delta} \tau_{\geqslant\bullet}(E^{\otimes n+1} \otimes X) \\
&\simeq \mathrm{AF}_{E}(X).
\end{aligned}$$

Finally, to see the claimed naturality let $f\colon E \to E'$ be a map of commutative algebra objects in spectra, such that E and E' are Adams-type. By Theorem III.4.2.10 we have an equivalence $f_!\nu_E X \simeq \nu_{E'} X$ for all spectra X. Therefore, the diagram

$$\begin{tikzcd}
\mathrm{Mod}_{\nu_E E}(\mathrm{Syn}_E) \arrow[r, "f_!"] \arrow[d] & \mathrm{Mod}_{\nu_{E'} E}(\mathrm{Syn}_{E'}) \arrow[d] \\
\mathrm{Mod}_{\nu_E E'}(\mathrm{Syn}_E) \arrow[r, "f_!"] & \mathrm{Mod}_{\nu_{E'} E'}(\mathrm{Syn}_{E'}).
\end{tikzcd}$$

commutes and the lower horizontal map is an equivalence by Corollary III.4.2.8, since $\nu_E E'$ is a $\nu_E E$-algebra. In particular, arguing as above we gain a natural transformation

$$\tilde{f}^{\bullet}_{!}\colon \mathrm{Mod}_{\nu_E E^{\otimes \bullet+1}}(\mathrm{Syn}_E) \xrightarrow{f_!} \mathrm{Mod}_{\nu_E (E')^{\otimes \bullet+1}}(\mathrm{Syn}_E) \xrightarrow{\sim} \mathrm{Mod}_{\nu_{E'} (E')^{\otimes \bullet+1}}(\mathrm{Syn}_{E'})$$

of diagrams $\Delta \to \mathrm{CAlg}(\mathrm{Pr}^L)$. Furthermore, the fact that diagram above commutes, tells us that the natural transformation

$$\tilde{f}_!\mu_E \to \mu_{E'} f_!$$

is an equivalence. This completes the proof. □

Proposition III.4.2.12. *The functor $\mu\colon \mathrm{Syn}_E \to \mathrm{QCoh}(\mathcal{M}_E^\tau)$ is fully faithful on the full subcategory of synthetic spectra, which are νE-nilpotent complete.*

Proof. By construction the functor μ factors as a composite

$$\mathrm{Syn}_E \xrightarrow{p} \lim_{[n]\in\Delta} \mathrm{Mod}_{\nu E^{\otimes n+1}}(\mathrm{Syn}_E) \xrightarrow{\sim} \mathrm{QCoh}(\mathcal{M}_E^\tau),$$

where the latter is an equivalence. Therefore, it suffices to show that p is fully faithful on those synthetic spectra, which are νE-nilpotent complete. For any synthetic spectrum X, the map

$$X \to \lim_{[n]\in\Delta} \nu E^{\otimes n+1} \otimes X$$

is the counit of the adjunction $p \dashv p^R$, so X is νE-nilpotent complete if and only if the counit is an equivalence. It is a standard exercise in category theory to verify that p is fully faithful on the full subcategory where the counit is an equivalence. □

Proposition III.4.2.13 ([BHS23, Prop. A.13])**.** *For a spectrum X the following are equivalent.*

1. *The spectrum X is E-nilpotent complete.*
2. *The E-based synthetic spectrum νX is νE-nilpotent complete.*
3. *The E-based synthetic spectrum νX is τ-complete. That is, the is evident map $\nu E \to \lim_{[n]\in\Delta} C\tau^n \otimes \nu X$ is an equivalence.*

Remark III.4.2.14. Proposition III.4.2.12 tells us that the failure of μ to be fully faithful is equivalent to the failure of weak convergence of the E-based Adams spectral sequence. This provides a philosophical explanation to the difference of Syn_E and $\mathrm{QCoh}(\mathcal{M}_E^\tau)$. The category of E-based synthetic spectra contains $\mathcal{S}p$ as a full subcategory, and we like to imagine a synthetic spectrum X as a pair of $E, \tau^{-1}X$ of an Adams spectral sequence E which does not necessarily converge to the underlying spectrum of X, if one tries to replicate this analogy for $\mathrm{QCoh}(\mathcal{M}_E^\tau)$ the spectral sequence is forced to be weakly convergent.

Using the above we can now prove the following result suggested to us by William Balderrama.

Theorem III.4.2.15. If $E \in \mathrm{CAlg}(\mathcal{S}p)$ is Adams-type, then the functor

$$\mu\colon \mathrm{Syn}_E \to \mathrm{QCoh}(\mathcal{M}_E^\tau)$$

identifies $\mathrm{QCoh}(\mathcal{M}_E^\tau)$ with the left completion of Syn_E, with respect to the homological t-structure on Syn_E and the canonical t-structure on $\mathrm{QCoh}(\mathcal{M}_E^\tau)$.

Proof. Since $\mathcal{M}_E^\tau$ is a geometric filtered stack by Corollary III.3.1.29 and Proposition III.4.1.5 its t-structure is left complete by Theorem III.3.2.12. Hence, it suffices to show that $\mu\colon \mathrm{Syn}_E \to \mathrm{QCoh}(\mathcal{M}_E^\tau)$ restricts to an equivalence

$$\mathrm{Syn}_{E,\leqslant n} \xrightarrow{\sim} \mathrm{QCoh}(\mathcal{M}_E^\tau)_{\leqslant n}$$

for all $n \in \mathbb{Z}$. By Theorem III.4.2.11 μ is t-exact, so its right adjoint μ^R must be left t-exact. As a result, the adjunction $\mu \dashv \mu^R$ restricts to an adjunction on n-coconnective objects. To complete the proof we will now show the following two claims:

1. The functor $\mu\colon \mathrm{Syn}_{E,\leqslant n} \to \mathrm{QCoh}(\mathcal{M}_E^\tau)_{\leqslant n}$ is fully faithful.

2. The functor $\mu^R\colon \mathrm{QCoh}(\mathcal{M}_E^\tau)_{\leqslant n} \to \mathrm{Syn}_{E,\leqslant n}$ is conservative.

We first prove (1). Let $X \in \mathrm{Syn}_E$ be n-coconnective. It follows from Theorem III.4.2.11, that the unit $X \to \mu^R\mu X$ is equivalent to the νE-nilpotent completion of X. Now, since X is n-coconnective it is both $C\tau$-local and νE-local by [Pst22, Lemma 4.35] and [Pst22, Remark 2.17]. As a result we may reduce to showing that

$$C\tau \otimes X \to \lim_{[n]\in\Delta} E_*E^{\otimes_{E_*} n+1} \bigotimes_{E_*} (C\tau \otimes X)$$

is an equivalence in $D(\mathrm{Comod}_{E_*E}) \simeq \mathrm{QCoh}(\mathcal{M}_E^{d\heartsuit})$, by [Pst22, Theorem 4.54]. This follows from faithfully flat descent [HP24, Cor. 3.26] in Dirac Geometry.

We now prove (2). Consider $\mathcal{F} \in \mathrm{QCoh}(\mathcal{M}_E^\tau)_{\leqslant n}$ and suppose $\mu^R(\mathcal{F}) \simeq 0$. We have to show that $\mathcal{F} \simeq 0$. By construction μ is $\mathcal{S}p^{\mathrm{fil}}$-linear, so we obtain a functor $\mathrm{Mod}_{C\tau}(\mathrm{Syn}_E)_{\leqslant n} \to \mathrm{Mod}_{C\tau}(\mathrm{QCoh}(\mathcal{M}_E^\tau)) \simeq \mathrm{QCoh}(\mathcal{M}_E^{d\heartsuit})$, which is equivalent to the inclusion of n-coconnective objects by [Pst22, Remark 2.17], [Pst22, Lemma 4.35], and [Pst22, Theorem 4.54][1]. In particular, $\mu^R(\mathcal{F}) \simeq 0$ implies that $C\tau \otimes \mathcal{F}$. We finish the proof by showing that $\mathcal{F}$ is $C\tau$-local. It suffices to see that $\tau^{-1}\mathcal{F} \simeq 0$. Since pulling back along $p^\tau \colon \mathrm{Spec}^\tau E \to \mathcal{M}_E^\tau$ is conservative, $\mathcal{S}p^{\mathrm{fil}}$-linear and creates (co)connectivity, we may reduce to showing the analogous claim for coconnective $\tau_{\geqslant \bullet}E$-modules. This is however clear, since

$$\pi_n(\tau^{-1}M) \cong \pi_{n,k}(\mathrm{colim}_k M_k) \cong \mathrm{colim}_k \pi_n(M_k).$$

If M is coconnective in the Postnikov t-structure, the above colimit is eventually 0 and hence $\tau^{-1}M \simeq 0$. In particular, M is $C\tau$-local.

We conclude that $\mu \colon \mathrm{Syn}_{E,\leqslant n} \xrightarrow{\sim} \mathrm{QCoh}(\mathcal{M}_E^\tau)_{\leqslant n}$ is an equivalence. $\square$

Remark III.4.2.16. If $E \coloneqq E_n$ is height n Morava E-theory, then the smash product theorem for νE is known to fail. More precisely, the localization

$$L_{\nu E} \colon \mathrm{Syn}_E \to L_{\nu E}\,\mathrm{Syn}_E$$

[1] Pstrągowski considers the category $D(\mathrm{Comod}_{E_*E})$, but this is equivalent to $\mathrm{QCoh}(\mathcal{M}_E^{d\heartsuit})$, by the standard translation between Hopf algebroids and groupoids in stacks.

is not a smashing localization. We believe that the above result can be used to show the following weaker statement. For all E-based synthetic spectra X, the canonical map

$$L_{\nu E} X \to \lim_{[n] \in \Delta} \nu E^{\otimes n+1} \otimes X$$

is an equivalence. A standard argument shows that this happens, for X, if and only if

$$\nu E \otimes (\lim_{[n] \in \Delta} \nu E^{\otimes n+1} \otimes X) \to \lim_{[n] \in \Delta} \nu E \otimes \nu E^{\otimes n+1} \otimes X \simeq \nu E \otimes X,$$

is an equivalence. However, we have not been able to prove this. The above argument provides a proof in the case where X is n-coconnective for some $n \in \mathbb{Z}$. Thus, one can reduce to the case where X is a connective synthetic spectra, we have also not been able to prove this as of writing this thesis. We suspect that the connective case is easier, since their are non-trivial ∞-connective νE-local E-based synthetic spectra.

III.4.3 The even filtration

In this section we recall the construction of the *even filtration* of Hahn–Raksit–Wilson [HRW25].

Remark III.4.3.1. The association $X \mapsto \tau_{\geqslant 2\bullet} X$ refines to a lax symmetric monoidal functor

$$\mathcal{S}p \to \mathcal{S}p^{\mathrm{fil}},$$

such that if $X \in \mathcal{S}p$ is even, then $\mathrm{gr}_* \tau_{\geqslant 2\bullet} X \simeq \Sigma^{2*} \pi_{2*} X$. The proofs from Section III.2.1 carry over almost verbatim.

Proposition III.4.3.2. *The full subcategory* $(\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}})^{\mathrm{op}}$ *spanned by those commutative ring spectra, whose underlying spectrum is even, forms a subsite of* $(\mathrm{CAlg}(\mathcal{S}p))^{\mathrm{op}}$ *with the* π_**-flat topology.*

Proof. We have to check that if $A \to B$ is a cover with $A, B \in \mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}}$, then for all $A \to C$ with C even, $B \otimes_A C$ is even, and $C \to B \otimes_A C$ is faithfully flat.

Since the Künneth spectral sequence degenerates (since B is flat over A), we see that

$$\pi_* B \underset{\pi_* A}{\otimes} \pi_* C \cong \pi_*(B \underset{A}{\otimes} C).$$

In particular, $B \otimes_A C$ is even. It remains to be seen that $C \to B \otimes_A C$ is faithfully flat, but this follows immediately from combining Proposition III.2.2.15, Lemma III.2.2.20, and Proposition III.2.2.21. □

Corollary III.4.3.3. *The* π_**-flat topology on* $\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}}$ *is subcanonical.*

Proof. This follows immediately from Proposition III.4.3.2 and Corollary III.2.4.8. □

Definition III.4.3.4. The category of *even spectral stacks* $\mathrm{Stk}^{\mathrm{ev}}$ is the category of stacks on $\mathrm{Aff}^{\mathrm{ev}} \coloneqq (\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}})^{\mathrm{op}}$ with the π_*-flat topology.

Definition III.4.3.5 ([HRW25, Construction 2.1.3])**.** For a commutative ring spectrum $R \in \mathrm{CAlg}(\mathcal{S}p)$, the *even filtration* is the filtered commutative ring spectrum

$$\mathrm{fil}^{\mathrm{ev}}_* R \coloneqq \lim_{\substack{R \to S \\ S \text{ even}}} \tau_{\geqslant 2\bullet} S.$$

Remark III.4.3.6. Since complete filtered spectra are closed under limits in $\mathcal{S}p^{\mathrm{fil}}$, we immediately conclude that $\mathrm{fil}^{\mathrm{ev}}_* R$ is complete.

The even filtration satisfies excellent descent properties with respect to the following classes of maps.

Definition III.4.3.7 ([HRW25, Definition 2.2.15])**.** A map of commutative ring spectra $A \to B \in \mathrm{CAlg}(\mathcal{S}p)$ is *evenly (faithfully) flat* if for all $A \to C$ with C even, the tensor product $B \otimes_A C$ is even and (faithfully) flat over C.

Corollary III.4.3.8 ([HRW25, Cor. 2.2.17])**.** *If $R \to S$ is evenly faithfully flat, then the canonical map*

$$\mathrm{fil}^{\mathrm{ev}}_* R \xrightarrow{\sim} \lim_{[n]\in\Delta} \mathrm{fil}^{\mathrm{ev}}_*(S^{\otimes_R n+1})$$

is an equivalence.

Remark III.4.3.9. The underlying spectrum of $\mathrm{fil}^{\mathrm{ev}}_* R$ can be seen to be

$$\mathrm{colim}_{n\to-\infty} \mathrm{fil}^{\mathrm{ev}}_n R \simeq \lim_{\substack{R\to S \\ S \text{ even}}} S.$$

However, this limit need not be equivalent to R in general.

Proposition III.4.3.10 ([HRW25, Prop. 2.3.4 (1)])**.** *Let R be a commutative ring spectrum, and suppose there exists a map $R \to S$, which is evenly faithfully flat and satisfies descent, that is, the canonical map*

$$R \to \lim_{[n]\in\Delta} S^{\otimes_R n+1}$$

is an equivalence. If S is even, then the map

$$R \to \mathrm{colim}_{n\to-\infty} \mathrm{fil}^{\mathrm{ev}}_n R$$

is an equivalence.

Remark III.4.3.11. Forthcoming work of Robert Burklund and Achim Krause shows that for any connective commutative ring spectrum R, the underlying object of $\mathrm{fil}^{\mathrm{ev}}_* R$ is R. In fact, they show something stronger. They prove that for any connective commutative ring spectrum R, there exists an augmented cosimplicial diagram

$$R \longrightarrow R_0 \rightrightarrows R_1 \overset{\rightarrow}{\rightrightarrows} R_2 \overset{\rightrightarrows}{\rightrightarrows} \cdots$$

such that R_n is even for all $[n] \in \Delta$ and there is an equivalence

$$\mathrm{fil}^{\mathrm{ev}}_* R \xrightarrow{\sim} \lim_{[n]\in\Delta} \mathrm{fil}^{\mathrm{ev}}_* R_n.$$

Moreover, they show that any commutative ring spectrum R admits a map $R \to R^e$ to an even commutative ring. We say that R^e is an *even envelope* of R.

Theorem III.4.3.12 (Burklund–Krause)**.** If $R \in \mathrm{CAlg}(\mathcal{S}p)$ is κ-compact for some cardinal κ, then there exists an even envelope R^e, which is also κ-compact.

Using the work of Burklund and Krause we prove the following result.

Proposition III.4.3.13. *The map of sites* $\iota\colon \mathrm{Aff}^{\mathrm{ev}} \to \mathrm{Aff}$ *induces an adjunction:*

$$\mathrm{Shv}(\mathrm{Aff}^{\mathrm{ev}}_\kappa) \underset{\iota^*}{\overset{\iota_!}{\rightleftarrows}} \mathrm{Shv}(\mathrm{Aff}_\kappa).$$

Furthermore, the functor ι^* *preserves colimits and for any even commutative ring spectrum* A *we have an equivalence* $\iota^* \operatorname{Spec} A = \operatorname{Spec} A$.

Proof. The category $\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}}$ is accessible by [HRW25, Prop. 2.1.2], and if $\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}}_\kappa$ denotes the full subcategory of κ-compact objects, then $\mathrm{Aff}_\kappa^{\mathrm{ev}} := (\mathrm{CAlg}(\mathcal{S}p)_\kappa^{\mathrm{ev}})^{\mathrm{op}}$ forms a subsite of $\mathrm{Aff}^{\mathrm{ev}}$. In particular, we obtain an equivalence:

$$\operatorname{colim}_\kappa \mathrm{Shv}(\mathrm{Aff}_\kappa^{\mathrm{ev}}) \xrightarrow{\sim} \mathrm{Stk}^{\mathrm{ev}},$$

by [BDL25, Cor. 2.1.2.5]. So, since $\mathrm{CAlg}(\mathcal{S}p)$ is accessible, it suffices to define a functor

$$(-)^{\mathrm{e}}_\kappa\colon \mathrm{Shv}(\mathrm{Aff}_\kappa) \to \mathrm{Shv}(\mathrm{Aff}_\kappa^{\mathrm{ev}})$$

for all sufficiently large cardinals κ and verify that if $\kappa \leqslant \kappa'$, then the diagram

$$\begin{tikzcd}
\mathrm{Shv}(\mathrm{Aff}_\kappa) \arrow[r, "(-)^{\mathrm{e}}_\kappa"] \arrow[d] & \mathrm{Shv}(\mathrm{Aff}_\kappa^{\mathrm{ev}}) \arrow[d] \\
\mathrm{Shv}(\mathrm{Aff}_{\kappa'}) \arrow[r, "(-)^{\mathrm{e}}_{\kappa'}"] & \mathrm{Shv}(\mathrm{Aff}_{\kappa'}^{\mathrm{ev}})
\end{tikzcd}$$

commutes[1].

For all κ let $\iota_\kappa\colon \mathrm{Aff}_\kappa^{\mathrm{ev}} \hookrightarrow \mathrm{Aff}_\kappa$ denote the inclusion such that for all $\kappa \leqslant \kappa'$ we get a square of small sites

$$\begin{tikzcd}
\mathrm{Aff}_\kappa^{\mathrm{ev}} \arrow[r, "\iota_\kappa"] \arrow[d, "j^{\mathrm{ev}}"] & \mathrm{Aff}_\kappa \arrow[d, "j"] \\
\mathrm{Aff}_{\kappa'}^{\mathrm{ev}} \arrow[r, "\iota_{\kappa'}"] & \mathrm{Aff}_{\kappa'}\,.
\end{tikzcd}$$

Since these sites are small we get an adjunction

$$\mathrm{Shv}(\mathrm{Aff}_\kappa^{\mathrm{ev}}) \underset{\iota_\kappa^*}{\overset{(\iota_\kappa)_!}{\rightleftarrows}} \mathrm{Shv}(\mathrm{Aff}_\kappa)$$

[1]Technically, this is not sufficient to produce a natural transformation of diagrams, but it will be clear from the construction that this does define a natural transformation.

given by left Kan extension and restriction respectively. In particular, by passing to the mate, we obtain a lax natural transformation filling the square

$$\begin{tikzcd} \mathrm{Shv}(\mathrm{Aff}_\kappa) \arrow[r, "\iota_\kappa^*"] \arrow[d, "j_!"] & \mathrm{Shv}(\mathrm{Aff}_\kappa^{\mathrm{ev}}) \arrow[d, "j_!^{\mathrm{ev}}"] \\ \mathrm{Shv}(\mathrm{Aff}_{\kappa'}) \arrow[r, "\iota_{\kappa'}^*"] & \mathrm{Shv}(\mathrm{Aff}_{\kappa'}^{\mathrm{ev}}). \end{tikzcd}$$

Hence, it suffices to show that the natural transformation

$$j_!^{\mathrm{ev}} \iota_\kappa^* \to \iota_{\kappa'}^* j_!$$

is an equivalence. We will first show that the analogous Beck–Chevalley map filling the square

$$\begin{tikzcd} \mathcal{P}(\mathrm{Aff}_\kappa) \arrow[r, "\iota_\kappa^*"] \arrow[d, "j_!"] & \mathcal{P}(\mathrm{Aff}_\kappa^{\mathrm{ev}}) \arrow[d, "j_!^{\mathrm{ev}}"] \\ \mathcal{P}(\mathrm{Aff}_{\kappa'}) \arrow[r, "\iota_{\kappa'}^*"] & \mathcal{P}(\mathrm{Aff}_{\kappa'}^{\mathrm{ev}}). \end{tikzcd}$$

is an equivalence. We can check this pointwise so let $X \in \mathcal{P}(\mathrm{Aff}_\kappa)$ and consider $A \in (\mathrm{Aff}_{\kappa'}^{\mathrm{ev}})^{\mathrm{op}} = \mathrm{CAlg}(\mathcal{S}p)_{\kappa'}^{\mathrm{ev}}$. The source of the map is given by

$$\begin{aligned} j_!^{\mathrm{ev}} \iota_\kappa^* X(A) &\simeq \operatorname{colim}_{j^{\mathrm{ev}}(B) \to A} \iota_\kappa^* X(B) \\ &\simeq \operatorname{colim}_{j^{\mathrm{ev}}(B) \to A} X(\iota_\kappa B). \end{aligned}$$

Similarly, we see that the target is given by:

$$\begin{aligned} \iota_{\kappa'}^* j_! X(A) &\simeq j_! X(\iota_{\kappa'} A) \\ &\simeq \operatorname{colim}_{j(B) \to \iota_{\kappa'} A} X(B). \end{aligned}$$

Here the indexing categories for the colimit are defined via the cartesian squares

$$\begin{tikzcd} j^{\mathrm{ev}}_{/A} \arrow[r] \arrow[d] & \mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}}_{\kappa'/A} \arrow[d] \\ \mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}}_{\kappa} \arrow[r, "j^{\mathrm{ev}}"] & \mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}}_{\kappa'} \end{tikzcd}$$

and

$$\begin{tikzcd} j_{/A} \arrow[r] \arrow[d] & \mathrm{CAlg}(sp)_{\kappa'/A} \arrow[d] \\ \mathrm{CAlg}(\mathcal{S}p)_{\kappa} \arrow[r] & \mathrm{CAlg}(sp)_{\kappa'} \end{tikzcd}$$

respectively. Since A is even there is a comparison functor

$$f\colon j^{\mathrm{ev}}_{/A} \to j_{/A},$$

which, after taking the colimit agrees with the component of $j^{\mathrm{ev}}_{!}\iota^*_{\kappa} \to \iota^*_{\kappa'} j_!$ at A. We will show that f is cofinal. One can check explicitly that the diagram is cartesian

$$\begin{tikzcd} j^{\mathrm{ev}}_{/A} \arrow[r] \arrow[d] & j_{/A} \arrow[d] \\ \mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}}_{\kappa} \arrow[r] & \mathrm{CAlg}(\mathcal{S}p)_{\kappa} \end{tikzcd}$$

is cartesian. Thus, it suffices to show that $\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}}_{\kappa} \to \mathrm{CAlg}(\mathcal{S}p)_{\kappa}$ is cofinal and that $j_{/A} \to \mathrm{CAlg}(\mathcal{S}p)_{\kappa}$ is a left fibration. The latter is clear, since $\mathrm{CAlg}(\mathcal{S}p)_{\kappa'/A} \to \mathrm{CAlg}(\mathcal{S}p)_{\kappa}$ is a left fibration and left fibrations are closed under pullback. To see that $\iota_{\kappa}\colon \mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}}_{\kappa} \to \mathrm{CAlg}(\mathcal{S}p)_{\kappa}$ is cofinal, we will apply Quillen's theorem A. For any $A \in \mathrm{CAlg}(\mathcal{S}p)_{\kappa}$, we have to show that the category $\iota_{\kappa/A}$ is weakly contractible. We will show that it is filtered. Let $F\colon I \to \iota_{\kappa/A}$ be a diagram, with I

finite. We need to show that there exists a cone $F' \colon I^{\triangleright} \to \iota_{\kappa/A}$. For all cardinals κ' the category $\mathrm{CAlg}(\mathcal{S}p)_{\kappa'}$ has κ'-small colimits, in particular finite ones. So the same holds for any slice of $\mathrm{CAlg}(\mathcal{S}p)_{\kappa'}$. In particular, the composite

$$I \to \iota_{\kappa/A} \to \mathrm{CAlg}(\mathcal{S}p)_{\kappa/A}$$

admits a colimit. If we let F^e denote the even envelope of the colimit of the above composite, which exists and is κ-compact by the theorem of Burklund–Krause Theorem III.4.3.12, then this defines a cone of F in $\iota_{\kappa/A}$. In particular, $\iota_{\kappa/A}$ is filtered, and hence weakly contractible. It follows that ι_κ and thus $f \colon j^{\mathrm{ev}}_{/A} \to j_{/A}$ is cofinal.

The above shows that

$$\begin{tikzcd}
\mathcal{P}(\mathrm{Aff}_\kappa) \arrow[r, "\iota_\kappa^*"] \arrow[d, "j_!"] & \mathcal{P}(\mathrm{Aff}^{\mathrm{ev}}_\kappa) \arrow[d, "j^{\mathrm{ev}}_!"] \\
\mathcal{P}(\mathrm{Aff}_{\kappa'}) \arrow[r, "\iota_{\kappa'}^*"] & \mathcal{P}(\mathrm{Aff}^{\mathrm{ev}}_{\kappa'}).
\end{tikzcd}$$

commutes. We can therefore reduce to showing that the diagram

$$\begin{tikzcd}
\mathrm{Shv}(\mathrm{Aff}_\kappa) \arrow[r] \arrow[d] & \mathcal{P}(\mathrm{Aff}_\kappa) \arrow[r, "\iota_\kappa^*"] \arrow[d, "j_!"] & \mathcal{P}(\mathrm{Aff}^{\mathrm{ev}}_\kappa) \arrow[r] \arrow[d, "j^{\mathrm{ev}}_!"] & \mathrm{Shv}(\mathrm{Aff}^{\mathrm{ev}}_\kappa) \arrow[d] \\
\mathrm{Shv}(\mathrm{Aff}'_\kappa) \arrow[r] & \mathcal{P}(\mathrm{Aff}_{\kappa'}) \arrow[r, "\iota_{\kappa'}^*"] & \mathcal{P}(\mathrm{Aff}^{\mathrm{ev}}_{\kappa'}) \arrow[r] & \mathrm{Shv}(\mathrm{Aff}^{\mathrm{ev}}_{\kappa'})
\end{tikzcd}$$

commutes. The right square commutes by construction, it suffices to show that the Beck–Chevalley map filling the left square is sent to an equivalences in $\mathrm{Shv}\,\mathrm{Aff}^{\mathrm{ev}}_{\kappa'})$. This is however clear, since the Beck-Chevalley map filling the left square is exactly sheafification, and the middle horizontal maps are induced by precomposing with ι_κ and $\iota_{\kappa'}$ respectively, which are maps of sites and hence preserve sheaves.

Putting all of this together, we see from [AGH25, Cor. 5.2.12] that $\{(\iota_\kappa)_!\}_\kappa$ is left adjoint to $\{\iota_\kappa^*\}_\kappa$ in the 2-category of functors from cardinals to Cat and taking the colimit is a 2-functor, so we get an induced adjunction

$$\mathrm{Stk}^{\mathrm{ev}} \underset{\iota^*}{\overset{\iota_!}{\rightleftarrows}} \mathrm{Stk}$$

From the construction, it is clear that for all $A \in \mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}}$ we have an equivalence $\iota^* \operatorname{Spec} A = \operatorname{Spec} A$. It remains to be seen that $\iota^* \colon \mathrm{Stk} \to \mathrm{Stk}^{\mathrm{ev}}$ preserves colimits, but this is an immediate consequences of the way it has been constructed, since $\iota_\kappa^* \colon \mathrm{Shv}(\mathrm{Aff}_\kappa) \to \mathrm{Shv}(\mathrm{Aff}_\kappa^{\mathrm{ev}})$ preserves colimits for all κ. $\square$

Definition III.4.3.14. For any spectral stack X, we define its associated *even stack* X^{ev} by

$$X^{\mathrm{ev}} := \iota^* X.$$

If $X = \operatorname{Spec} R$ for some commutative ring spectrum R, we will write

$$\operatorname{Spev} R := (\operatorname{Spec} R)^{\mathrm{ev}}.$$

Definition III.4.3.15. An affine map of spectral stacks $f \colon X \to Y$ is *evenly flat*, if for all $\operatorname{Spec} R \to Y$ map $R \to S$ induced by $\operatorname{Spec} S \simeq X \times_Y \operatorname{Spec} R \to \operatorname{Spec} R$ is flat. We say that f is *evenly faithfully flat* if it is evenly flat and an effective epimorphism.

Corollary III.4.3.16. *If X is a geometric spectral stack, with an evenly flat cover $p \colon \operatorname{Spec} R \to X$ and R is even, then X^{ev} is a geometric even spectral stack.*

Proof. For a cover $p \colon \operatorname{Spec} R \to X$ as above, let $\operatorname{Spec} R^\bullet$ denote the Čech nerve of p. In this situation, R^n is even for all $[n] \in \mathbb{\Delta}$. By construction $(-)^{\mathrm{ev}}$ preserves limits and colimits. In particular, $\operatorname{Spec} R \simeq \operatorname{Spev} R \to X^{\mathrm{ev}}$ is an effective epimorphism. Furthermore, p^{ev} is flat, by the even version of Lemma III.3.1.14 since $\operatorname{Spec} R^1 \to \operatorname{Spec} R$ is flat, and R^1 is even. $\square$

Proposition III.4.3.17. *If $f\colon X \to Y$ is evenly (faithfully) flat, then the induced map $f^{\mathrm{ev}}\colon X^{\mathrm{ev}} \to Y^{\mathrm{ev}}$ is (faithfully) flat.*

Proof. Suppose $f\colon X \to Y$ is evenly flat, and consider $R \in \mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}}$ with a map $\operatorname{Spec} R \to Y^{\mathrm{ev}}$; by adjunction, this corresponds to a map $\operatorname{Spec} R \to Y$. In this situation the cartesian square

$$\begin{tikzcd} \operatorname{Spec} S \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] & Y \end{tikzcd}$$

induces a cartesian square

$$\begin{tikzcd} \operatorname{Spev} S \arrow[r] \arrow[d] & X^{\mathrm{ev}} \arrow[d] \\ \operatorname{Spev} R \arrow[r] & Y^{\mathrm{ev}} \end{tikzcd}$$

of even spectral stacks. By assumption R is even, so $\operatorname{Spev} R = \operatorname{Spec} R$, and it remains to be seen that S is even and $R \to S$ is π_*-flat. By construction $R \to S$ is evenly flat, so since R is even, this implies that $R \otimes_R S \simeq S$ is even and S is flat over R.

Finally, suppose f is evenly faithfully flat, then we have to show that $f^{\mathrm{ev}}\colon X^{\mathrm{ev}} \to Y^{\mathrm{ev}}$ is faithfully flat. Indeed, by the same argument as in Corollary III.4.3.16, $(-)^{\mathrm{ev}}$ preserves effective epimorphisms, so f^{ev} is an effective epimorphism and the above argument shows that it is flat. □

Corollary III.4.3.18. *If $R \to S$ is an evenly faithfully flat map of commutative ring spectra and S is even, then* $\operatorname{Spev} R$ *is a geometric even spectral stack.*

III.4.4 The filtered even stack

Building on the results of Section III.4.3, we will now construct a functor

$$\mathrm{Stk}^{\mathrm{ev}} \to \mathrm{FStack},$$

which on even affines is given by $\operatorname{Spec} R \mapsto \operatorname{Spec} \tau_{\geqslant 2\bullet} R$ and whose coherent cohomology encodes the even filtration of Hahn–Raksit–Wilson. In order to construct this functor we will need a short discussion on *even* filtered spectra; a more in-depth discussion can be found in [Nig25, Section 5.3.1].

Definition III.4.4.1 ([Nig25, Definition 5.15])**.** A filtered spectrum X is *even* if the map

$$X_{2n} \to X_{2n-1}$$

is an equivalence for all $n \in \mathbb{Z}$. We denote the full subcategory of even filtered spectra by $\mathcal{S}p^{\mathrm{fil}}_{\mathrm{ev}}$.

Theorem III.4.4.2 ([Nig25, Prop. 5.17])**.** The category of even filtered spectra is a presentable symmetric monoidal subcategory of $\mathcal{S}p^{\mathrm{fil}}$. Furthermore, the inclusion

$$\mathcal{S}p^{\mathrm{fil}}_{\mathrm{ev}} \to \mathcal{S}p^{\mathrm{fil}}$$

preserves both limits and colimits. In particular, it admits both a left and a right adjoint.

Example III.4.4.3. If X is an even spectrum, then $\tau_{\geqslant\bullet} X$ is an even filtered spectrum.

Remark III.4.4.4 ([Nig25, Remark 5.18])**.** One can show that the right adjoint of the inclusion $\mathcal{S}p^{\mathrm{fil}}_{\mathrm{ev}} \to \mathcal{S}p^{\mathrm{fil}}$ sends a filtered spectrum X to the filtered spectrum

$$\cdots \to X_2 \xrightarrow{\mathrm{id}} X_2 \to X_0 \xrightarrow{\mathrm{id}} X_0 \to X_{-2} \to \dots$$

Here X_0 sits in degree 0 and -1.

Remark III.4.4.5 ([Nig25, Remark 5.19])**.** Precomposing with $2\colon \mathbb{Z}^{\mathrm{op}} \to \mathbb{Z}^{\mathrm{op}}$ induces a symmetric monoidal functor

$$\mathcal{S}p^{\mathrm{fil}} \to \mathcal{S}p^{\mathrm{fil}}$$

whose restriction to $\mathcal{S}p^{\mathrm{fil}}_{\mathrm{ev}}$ is an equivalence. The functor is given by sending an even filtered spectrum $X_\bullet$ to $X_{2\bullet}$. We will denote this functor by $\rho\colon \mathcal{S}p^{\mathrm{fil}}_{\mathrm{ev}} \xrightarrow{\sim} \mathcal{S}p^{\mathrm{fil}}$.

Remark III.4.4.6. The functor $\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}} \to \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})$ sending R to $\tau_{\geqslant 2\bullet}R$ can be written as the composite

$$\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}} \xrightarrow{\tau_{\geqslant\bullet}} \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\mathrm{ev},\geqslant 0}) \xrightarrow{\rho} \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}).$$

Note that, for any spectrum X, the double speed Postnikov tower $\tau_{\geqslant 2\bullet}$ is not connective in the Postnikov t-structure on filtered spectra, since for all n the spectrum $\tau_{\geqslant 2n}X$ is $2n$-connective instead of n-connective, which for $n<0$, where $2n<n$ is not in agreement with the definition Postnikov connective. It is instead connective for the double speed Postnikov t-structure. For this reason $\mathrm{Spec}\,\tau_{\geqslant 2\bullet}R$ will denote the restricted Yoneda embedding, in practice this makes little difference, since the functor

$$\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\mathrm{ev}})_{\tau_{\geqslant\bullet}R/} \to \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})_{\tau_{\geqslant 2\bullet}R/},$$

induced by ρ, is final[1] for all $R \in \mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}}$. In fact, this holds for all even filtered ring spectra.

Theorem III.4.4.7. The functor

$$\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}} \to \mathrm{FStack}^{\mathrm{op}}$$

[1] This holds for all left adjoints, in particular equivalences.

given by sending R to $\operatorname{Spec} \tau_{\geqslant 2\bullet} R$ is a sheaf for the π_*-flat topology. In particular, there exists a colimit-preserving functor

$$(-)^{\tau-\mathrm{ev}} \colon \mathrm{Stk}^{\mathrm{ev}} \to \mathrm{FStack}$$

such that $(\operatorname{Spec} R)^{\tau-\mathrm{ev}} = \operatorname{Spec} \tau_{\geqslant 2\bullet} R$.

Proof. We have to show that the functor $\operatorname{Spec} \tau_{\geqslant 2\bullet} \colon \mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}} \to \mathrm{FStack}^{\mathrm{op}}$ satisfies the following two conditions:

1. It preserves finite products.
2. For all faithfully π_*-flat maps $R \to S$, the canonical map

$$\operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} \operatorname{Spec}(\tau_{\geqslant 2\bullet}(S^{\otimes_R n+1})) \to \operatorname{Spec} \tau_{\geqslant 2\bullet} R$$

   is an equivalence.

It is clear that it preserves products, so we will focus on (2).

Let $R \to S$ be a faithfully π_*-flat map of even commutative ring spectra, then by faithfully flat descent Theorem III.2.4.3 and Lemma III.2.2.20, we have equivalences

$$\begin{aligned}\tau_{\geqslant\bullet} R &\simeq \lim_{[n]\in\Delta} \tau_{\geqslant\bullet}(S)^{\otimes_{\tau_{\geqslant\bullet}(R)} n+1} \\ &\simeq \lim_{[n]\in\Delta} \tau_{\geqslant\bullet}(S^{\otimes_R n+1}),\end{aligned}$$

in $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\mathrm{ev},\geqslant 0})$. Since $\rho\colon \mathcal{S}p^{\mathrm{fil}}_{\mathrm{ev}} \xrightarrow{\sim} \mathcal{S}p^{\mathrm{fil}}$ is a symmetric monoidal equivalence we see that there are equivalences

$$\begin{aligned}\tau_{\geqslant 2\bullet} R &\simeq \lim_{[n]\in\Delta} \tau_{\geqslant 2\bullet}(S)^{\otimes_{\tau_{\geqslant 2\bullet}(R)} n+1} \\ &\simeq \lim_{[n]\in\Delta} \tau_{\geqslant 2\bullet}(S^{\otimes_R n+1}),\end{aligned}$$

in $\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})$. In particular, if we consider the Yoneda embedding

$$よ\colon \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})^{\mathrm{op}} \to \mathcal{P}(\mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0})^{\mathrm{op}})$$

we see that $よ(\tau_{\geqslant 2\bullet} S) \to よ(\tau_{\geqslant 2\bullet} R)$ is an effective epimorphism. Hence, after sheafification, we see that

$$\operatorname{Spec} \tau_{\geqslant 2\bullet} S \to \operatorname{Spec} \tau_{\geqslant 2\bullet} R$$

is an effective epimorphism in FStack. In particular, we obtain an equivalence

$$\begin{aligned} \operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} \operatorname{Spec}(\tau_{\geqslant 2\bullet}(S^{\otimes_R n+1})) &\simeq \operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} \operatorname{Spec}(\tau_{\geqslant 2\bullet} S^{\otimes_{\tau_{\geqslant 2\bullet} R}\, n+1}) \\ &\simeq \operatorname{Spec} \tau_{\geqslant 2\bullet} R. \end{aligned} \qquad \square$$

Definition III.4.4.8. If X is an even spectral stack, then we denote the *associated filtered even stack* as the image of X under $(-)^{\tau-\mathrm{ev}}$. If X is a spectral stack, then we will abuse notation and write

$$X^{\tau-\mathrm{ev}} := (X^{\mathrm{ev}})^{\tau-\mathrm{ev}},$$

for the filtered even stack associated to the even spectral stack X^{ev}.

Remark III.4.4.9. One would expect that $(-)^{\tau-\mathrm{ev}}$ would be induced by a map of sites $\mathrm{CAlg}(\mathcal{S}p)^{\mathrm{ev}} \to \mathrm{CAlg}(\mathcal{S}p^{\mathrm{fil}})$ given by the double speed Postnikov tower. Unfortunately, we have not been able to show that this is a map of sites.

Proposition III.4.4.10. *Let R be a commutative ring spectrum. If $R \to S$ is evenly faithfully flat and S is even, then we have an equivalence of filtered ring spectra*

$$\Gamma((\operatorname{Spev} R)^{\tau-\mathrm{ev}}) \simeq \operatorname{fil}^{\mathrm{ev}}_* R.$$

Proof. It follows from Proposition III.4.3.17 that $\operatorname{Spev} R$ is a geometric even spectral stack with such that $\operatorname{Spec} S \to \operatorname{Spev} R$ is a flat cover. In particular, if $\operatorname{Spec} S^\bullet$ is the Čech nerve of $\operatorname{Spec} S \to \operatorname{Spev} R$, then $S^k = S^{\otimes_R k}$ is even all k, so we obtain an equivalence

$$\operatorname{colim}_{[n]\in\Delta} \operatorname{Spec} \tau_{\geqslant 2\bullet} S^n \xrightarrow{\simeq} (\operatorname{Spev} R)^{\tau-\mathrm{ev}}.$$

It follows from Corollary III.3.4.6 that

$$\Gamma((\operatorname{Spev} R)^{\tau-\mathrm{ev}}) \simeq \lim_{[n]\in\Delta} \tau_{\geqslant 2\bullet} S^n \simeq \operatorname{fil}^{\mathrm{ev}}_* R.$$

Here the last equivalence is Corollary III.4.3.8. □

Definition III.4.4.11. If X is a spectral stack, then the *even filtration* of X, is the filtered spectrum

$$\operatorname{fil}^{\mathrm{ev}}_* X \coloneqq \Gamma(X^{\tau-\mathrm{ev}})$$

given by the cohomology of the filtered even stack associated to X.

Example III.4.4.12. Consider $\mathcal{M}_{\mathrm{MU}}$ the descent stack of $\operatorname{Spec} \mathrm{MU}$, then the even filtration $\mathcal{M}_{\mathrm{MU}}$ agrees with the double speed Adams filtration of the sphere. That is there is an equivalence

$$\operatorname{fil}^{\mathrm{ev}}_* \mathcal{M}_{\mathrm{MU}} \simeq \lim_{[n]\in\Delta} \tau_{\geqslant 2\bullet} \mathrm{MU}^{\otimes n+1}.$$

Chapter III.5

Appendix

A.1 t-structure flat topologies

In this appendix we establish the results of Section III.2.2 in the general setting of a presentably symmetric monoidal stable ∞-category $\mathcal{C}$ with a separated and right complete t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$, which is compatible with filtered colimits and the symmetric monoidal structure. The main observation is that there is always a good notion of a *flat topology* on $\mathrm{CAlg}(\mathcal{C}_{\geqslant 0})^{\mathrm{op}}$ and that the main computational tools established for the flat topology on ring spectra in [Lur17b, Section 7] exist in this case. For the remainder of this section we will fix $\mathcal{C}$ as above. The functor $\tau_{\geqslant \bullet}\colon \mathcal{C} \to \mathcal{C}^{\mathrm{fil}}$ will denote the Postnikov tower for the fixed t-structure, which can be seen to be lax symmetric monoidal and takes values in $\mathcal{C}^{\mathrm{fil}}_{\geqslant 0}$, where $\mathcal{C}^{\mathrm{fil}}$ is given the Postnikov t-structure. That is, a filtered object X is connective if and only if $X_n \in \mathcal{C}_{\geqslant n}$ for all $n \in \mathbb{Z}$. Furthermore, we let $\pi_k\colon \mathcal{C} \to \mathcal{C}^{\heartsuit}$ denote the k-th t-structure homotopy group. This section is heavily inspired by work of Jacob Lurie in [Lur17b, Section 7] and [Lurb, Appendix B.6 and D.6], and we expect that most of the results are known to experts in the field.

In the final two sections we will quickly review parts of the theory of Dirac geometry as introduced by Hesselholt and Pstrągowski in [HP25] and [HP24] and the theory of stacks on a site as developed by Balderrama, Davies and Linskens in [BDL25, Part 2]. Most of the results of this section are in no way due to the author of this paper, and the ones that are carry very little interest without the context of the others.

Definition A.1.1. Let $R \in \mathrm{CAlg}(\mathcal{C}^\heartsuit)$ be a commutative algebra and $M \in \mathrm{Mod}_R(\mathcal{C}^\heartsuit)$ be an R-module. We say that M is *flat*[1] if the functor

$$M \underset{R}{\otimes} -\colon \mathrm{Mod}_R(\mathcal{C}^\heartsuit) \to \mathrm{Mod}_R(\mathcal{C}^\heartsuit),$$

defined by the symmetric monoidal structure on $\mathcal{C}^\heartsuit$ induced by the one on $\mathcal{C}$, is an exact functor of abelian categories.

Definition A.1.2. Let $R \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra. An R-module $M \in \mathrm{Mod}_R(\mathcal{C})$ is *t-structure flat*[2] if $\pi_0 M$ is flat over $\pi_0 R$ and the canonical map

$$\pi_k R \underset{\pi_0 R}{\otimes} \pi_0 M \to \pi_k M$$

is an equivalence for all $k \in \mathbb{Z}$. A map of commutative algebras $R \to S$ is *t-structure flat* if S is flat as an R-module.

Notation A.1.3. Let $\mathcal{C}$ be as above. For any filtered object $X \in \mathcal{C}^{\mathrm{fil}}$ we define the *(p,q)-th homotopy group* of X to be

$$\pi_{p,q} X \coloneqq \pi_p(X_q).$$

For any graded object $Y \in \mathcal{C}^{\mathrm{gr}}$ we use the same notation.

[1] This is a slight abuse of terminology, however since the term flat is used consistently in this paper, we do not expect this will cause any problems.

[2] We will often abuse terminology and just say flat, this will not cause any problems since if R is discrete, then this definition coincides with Definition A.1.1

Proposition A.1.4. *Let R be a commutative algebra in $\mathcal{C}$ and let M and N be R-modules. In this situation, there is a spectral sequence computing $\pi_*(M \otimes_R N)$ with E_1-page*

$$E_1^{p,q} = \pi_{p,q}(\Sigma^*\pi_*M \underset{\Sigma^*\pi_*R}{\otimes} \Sigma^*\pi_*N) \Rightarrow \pi_p(M \underset{R}{\otimes} N).$$

Furthermore, the spectral sequence is convergent.

Proof. By [Lur17b, Definition 1.2.2.9, Prop. 1.2.2.14] we have to produce a complete filtered object $X_\bullet$ in $\mathcal{C}$, whose associated graded has bigraded homotopy groups given by

$$\pi_{p,q}\operatorname{gr}_* X_\bullet \cong \pi_{p,q}(\Sigma^*\pi_*M \underset{\Sigma^*\pi_*R}{\otimes} \Sigma^*\pi_*N),$$

and whose colimit is $M \otimes_R N$. Since $\operatorname{gr}_*\colon \mathcal{C}^{\mathrm{fil}} \to \mathcal{C}^{\mathrm{gr}}$ is symmetric monoidal the filtered object can be chosen to be $\tau_{\geqslant\bullet}M \otimes_{\tau_{\geqslant\bullet}R} \tau_{\geqslant\bullet}N$. $\square$

Example A.1.5. Let $\mathcal{C} = \mathcal{S}p$ and let $R \in \mathrm{CAlg}(\mathcal{S}p)$ be a ring spectrum. If M and N are R-module spectra, then the spectral sequence constructed in Proposition A.1.4 has E_1-page given by the bigraded homotopy groups of the graded spectrum

$$\Sigma^*\pi_*M \underset{\Sigma^*\pi_*R}{\otimes} \Sigma^*\pi_*N \cong \Sigma^*(\pi_*M \underset{\pi_*R}{\otimes} \pi_*N).$$

Since π_*R is discrete as graded spectrum it follows from [HP25, Prop. 3.4] that the tensor product

$$\pi_*M \underset{\pi_*R}{\otimes} \pi_*N$$

may equivalently be computed as the derived tensor product in the derived category $D(\mathrm{Mod}_{\pi_*R}(\mathrm{Ab}^{\mathrm{gr}}))$. In particular, we obtain an identification

$$\pi_{p,q}(\Sigma^*\pi_*M \underset{\Sigma^*\pi_*R}{\otimes} \Sigma^*\pi_*N) \cong \mathrm{Tor}^{p,q}_{\pi_*R}(\pi_*M, \pi_*N).$$

In order for this to match up with the usual grading for the Künneth spectral sequence starting at the E_2-page, one can use the convention

$$'E_2^{p,q} = E_1^{-q,p+2q}.$$

See [Ant24] for a reference.

Remark A.1.6. Here the shearing functor is constructed as in Construction III.2.1.28.

In the above proposition we obtain a spectral sequence whose E_1-page is given by the bigraded homotopy groups of the graded object

$$\Sigma^*\pi_*M \underset{\Sigma^*\pi_*R}{\otimes} \Sigma^*\pi_*N.$$

Here, the tensor product is computed in the symmetric monoidal ∞-category $\mathcal{C}^{\mathrm{gr}}$, despite all the objects being discrete. As a consequence, we need to obtain better control of this tensor product, in terms of the homological algebra of $\mathcal{C}^{\mathrm{gr}\heartsuit} \simeq (\mathcal{C}^{\heartsuit})^{\mathrm{gr}}$.

Definition A.1.7. Let R, M and N be as in Proposition A.1.4, we define the *Künneth spectral sequence* of R, M and N to be the spectral sequence constructed in Proposition A.1.4.

Remark A.1.8. Some authors like to call the above spectral sequence the Tor-*spectral sequence*, we do not distinguish between names, since one can deduce one from the other.

Proposition A.1.9. *Let R be a commutative algebra in $\mathcal{C}$ and let M be an R-module. If M is flat, then $\tau_{\geqslant\bullet}M$ and $\Sigma^*\pi_*M$ are flat in* $\mathrm{Mod}_{\tau_{\geqslant\bullet}R}(\mathcal{C}^{\mathrm{fil}})$ *and* $\mathrm{Mod}_{\Sigma^*\pi_*R}(\mathcal{C}^{\mathrm{gr}})$, *respectively.*

Proof. For the first part we follow the proof of Proposition III.2.2.21. If M is a flat R-module, then for every $N \in \mathcal{C}^{\mathrm{gr}\heartsuit}$ we have equivalences

$$\begin{aligned}\pi_0^P\tau_{\geqslant\bullet}M \underset{\pi_0^P\tau_{\geqslant\bullet}R}{\otimes} N &\cong \pi_*M \underset{\pi_*R}{\otimes} N\\ &\cong \pi_0M \underset{\pi_0R}{\otimes} N.\end{aligned}$$

By assumption the functor

$$\pi_0 M \underset{\pi_0 R}{\otimes} -\colon \operatorname{Mod}_{\pi_* R}(\mathcal{C}^{\mathrm{gr}\heartsuit}) \to \operatorname{Mod}_{\pi_* R}(\mathcal{C}^{\mathrm{gr}\heartsuit})$$

is exact. In particular $\pi_0^P \tau_{\geqslant \bullet} M$ is flat over $\pi_0^P \tau_{\geqslant \bullet} R$. It remains to see that

$$\pi_k^P \tau_{\geqslant \bullet} R \underset{\pi_0^P \tau_{\geqslant \bullet} R}{\otimes} \pi_0^P \tau_{\geqslant \bullet} M \to \pi_k^P \tau_{\geqslant \bullet} M$$

is an equivalence for every integer k. If $k < 0$, then this is trivially true, since both sides are 0, so we may assume that $k \geqslant 0$. In the case that $k \geqslant 0$, then we have

$$\begin{aligned} \pi_k^P \tau_{\geqslant \bullet} R \underset{\pi_0^P \tau_{\geqslant \bullet} R}{\otimes} \pi_0^P \tau_{\geqslant \bullet} M &\cong \pi_{*+k} R \underset{\pi_* R}{\otimes} \pi_* M \\ &\cong \pi_{*+k} M. \end{aligned}$$

It is easy to verify that the canonical map is an equivalence.

For the case of $\Sigma^* \pi_* M$, note that for any connective filtered object in $\mathcal{C}^{\mathrm{fil}}$ - in this case $\tau_{\geqslant \bullet} M$ - taking the associated graded is the same as 0-truncation. So it suffices to check that $\pi_* M$ is flat over $\pi_* R$, which we checked above. $\square$

It is helpful for the remaining proofs of this section to know that when $R \in \operatorname{CAlg}(\mathcal{C})$ is connective, then $\operatorname{Mod}_R(\mathcal{C})$ admits a t-structure determined by the one on $\mathcal{C}$. In order to prove this, we first prove the following lemma.

Lemma A.1.10. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and R be a commutative algebra object in $\mathcal{C}$. If $M, N \in \operatorname{Mod}_R(\mathcal{C})$, then there is an equivalence*

$$\operatorname{Map}_{\operatorname{Mod}_R(\mathcal{C})}(M, N) \simeq \lim_{k \in \Delta} \operatorname{Map}_{\mathcal{C}}(R^{\otimes k} \otimes M, N).$$

Proof. Let U denote the forgetful functor $\mathrm{Mod}_R(\mathcal{C}) \to \mathcal{C}$. Consider the simplicial R-module $R^{\otimes *+1} \otimes U(M)$ coming from the adjunction $R \otimes - \dashv U$. Since M is an R-module, this admits an extra degeneracy so we obtain an equivalence $\operatorname{colim}_{k \in \Delta^{\mathrm{op}}} R^{\otimes k+1} \otimes U(M) \simeq M$. It follows that

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Mod}_R(\mathcal{C})}(M, N) &\simeq \mathrm{Map}_{\mathrm{Mod}_R(\mathcal{C})}(\operatorname{colim}_{k \in \Delta^{\mathrm{op}}} R^{\otimes k+1} \otimes U(M), N) \\
&\simeq \lim_{k \in \Delta} \mathrm{Map}_{\mathrm{Mod}_R(\mathcal{C})}(R^{\otimes k+1} \otimes U(M), N) \\
&\simeq \lim_{k \in \Delta} \mathrm{Map}_{\mathcal{C}}(R^{\otimes k} \otimes U(M), U(N)).
\end{aligned}$$

□

Proposition A.1.11. *Let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞-category with a right complete t-structure, which is compatible with the symmetric monoidal structure. If $R \in \mathrm{CAlg}(\mathcal{C})$ is connective, then $\mathrm{Mod}_R(\mathcal{C})$ admits a right complete t-structure $(\mathrm{Mod}_R(\mathcal{C})_{\geqslant 0}, \mathrm{Mod}_R(\mathcal{C})_{\leqslant 0})$, such that the following hold:*

1. *For every $k \in \mathbb{Z}$ the diagram*

$$\begin{tikzcd}
\mathrm{Mod}_R(\mathcal{C}) \arrow[r, "\tau_{\geqslant k}"] \arrow[d] & \mathrm{Mod}_R(\mathcal{C}) \arrow[d] \\
\mathcal{C} \arrow[r, "\tau_{\geqslant k}"] & \mathcal{C}
\end{tikzcd}$$

   commutes.

2. *If the t-structure on $\mathcal{C}$ is left complete, then so is the induced t-structure on $\mathrm{Mod}_R(\mathcal{C})$.*

3. *The heart is given by $\mathrm{Mod}_{\pi_0 R}(\mathcal{C}^{\heartsuit})$.*

4. *The t-structure is compatible with the symmetric monoidal structure on $\mathrm{Mod}_R(\mathcal{C})$ given by the relative tensor product of R-modules.*

5. *If the t-structure on $\mathcal{C}$ is compatible with filtered colimits, then so is the one on* $\mathrm{Mod}_R(\mathcal{C})$.

6. *An R-module M is coconnective if and only if its underlying image is.*

Proof. Suppose R is as above. Let $U\colon \mathrm{Mod}_R(\mathcal{C}) \to \mathcal{C}$ denote the forgetful functor. By [Lur17b, Prop. 1.4.4.11 (1)] there is a t-structure $(\mathrm{Mod}_R(\mathcal{C})_{\geqslant 0}, \mathrm{Mod}_R(\mathcal{C})_{\leqslant 0})$ on $\mathrm{Mod}_R(\mathcal{C})$ where the connective part is given by $\mathrm{Mod}_R(\mathcal{C})_{\geqslant 0} = U^{-1}(\mathrm{Mod}_R(\mathcal{C})_{\geqslant 0})$. We will now show that $\mathrm{Mod}_R(\mathcal{C})_{\leqslant 0} = U^{-1}(\mathcal{C}_{\leqslant 0})$. Suppose $M \in \mathcal{C}_{\leqslant 0}$, we have to show that for any connective $X \in \mathcal{C}$

$$\mathrm{Map}_{\mathcal{C}}(X, \Omega U(M)) \simeq \mathrm{Map}_{\mathcal{C}}(X, U(\Omega M)) \simeq 0.$$

Since the t-structure on $\mathcal{C}$ is compatible with the symmetric monoidal structure we know that $R \otimes X$ is connective, hence

$$\mathrm{Map}_{\mathcal{C}}(X, \Omega U(M)) \simeq \mathrm{Map}_{\mathrm{Mod}_R(\mathcal{C})}(R \otimes X, \Omega M) \simeq 0.$$

Here, the last equivalence follows from our assumption that $U(M) \in \mathcal{C}_{\leqslant 0}$. In particular, $M \in \mathrm{Mod}_R(\mathcal{C})_{\leqslant 0}$. Suppose now that $N \in \mathrm{Mod}_R(\mathcal{C})_{\leqslant 0}$, in this case we have to show that for every $X \in \mathrm{Mod}_R(\mathcal{C})_{\geqslant 0}$, the mapping space $\mathrm{Map}_{\mathrm{Mod}_R(\mathcal{C})}(X, \Omega N)$ vanishes. It follows from Lemma A.1.10 that we have an equivalence

$$\mathrm{Map}_{\mathrm{Mod}_R(\mathcal{C})}(X, \Omega N) \simeq \lim_{k \in \mathbb{\Delta}} \mathrm{Map}_{\mathcal{C}}(R^{\otimes k} \otimes U(X), \Omega U(N)).$$

Now, to see that the right hand side vanishes it suffices to see that for every k the mapping space $\mathrm{Map}_{\mathcal{C}}(R^{\otimes k} \otimes U(M), \Omega U(N))$ is contractible. By assumption $U(M)$ is connective and thus by Lemma III.2.1.25 we see that $R^{\otimes k} \otimes U(M)$ is connective for every k. Similarly, $U(N)$ is coconnective by assumption, hence it follows from our assumption that $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ is a t-structure that

$$\mathrm{Map}_{\mathcal{C}}(R^{\otimes k} \otimes U(M), U(N)) \simeq 0. \qquad \square$$

Remark A.1.12. The existence statement and parts (1)-(5) are already in the literature cf. [AN21, Prop.A.15]. Part (6) is implicitly mentioned in their proof, but we choose to highlight it here since it is relevant to parts of our proofs.

Remark A.1.13. The statement in Proposition A.1.11 is very reminiscent of [Lur17b, Prop. 7.1.1.13] and [Lur17b, Lemma 7.1.3.10], except we do not recover the characterization of the connective part that Lurie makes. However, the connective part can be characterized as generated under colimits by objects of the form $R \otimes X$, where $X \in \mathcal{C}$ is connective.

Remark A.1.14. Parts of Proposition A.1.11 also hold for any commutative algebra object $R \in \mathrm{CAlg}(\mathcal{C})$. However, if R is not connective, the t-structure often fails to be interesting. For instance if $\mathcal{C} = \mathcal{S}p$ and $R = \mathrm{KU}$ is non-connective complex K-theory, then $\mathrm{Mod}_{\mathrm{KU}}(\mathcal{S}p)_{\geqslant 0} \simeq 0$. This can be seen, by noticing that for any KU-module M the Bott element $\beta \in \pi_{-2}\mathrm{KU}$ induces an equivalence

$$M \xrightarrow{\beta} \Sigma^{-2} M.$$

It follows that if the underlying spectrum of M is connective, then it must be 0.

Definition A.1.15. Let R be connective commutative algebra in $\mathcal{C}$ and let M and N be R-modules. We define the *Atiyah–Hirzebruch spectral sequence* of R, M and N to be the spectral sequence associated to the filtered object in $\mathrm{Mod}_R(\mathcal{C})$ given by

$$\mathrm{AH}_R(M, N) := (\tau_{\geqslant \bullet} M) \underset{R}{\otimes} N.$$

Proposition A.1.16. *Let R be a connective commutative algebra in $\mathcal{C}$ and let M and N be R-modules. Then the Atiyah–Hirzebruch spectral* *sequence of R, M and N has signature*

$$E_1^{p,q} = \pi_p(\Sigma^q \pi_q M \underset{R}{\otimes} N) \Rightarrow \pi_p(M \underset{R}{\otimes} N).$$

Furthermore, if N is bounded below, then $\mathrm{AH}_R(M, N)$ is a complete filtered object, so the spectral sequence is convergent.

Proof. By construction the E_1-page is given by

$$\begin{aligned}\pi_{p,q}(\Sigma^* \pi_* M \underset{R}{\otimes} N) &\cong \pi_p(\Sigma^q \pi_q M \underset{R}{\otimes} N)\\ &\cong \pi_p(\Sigma^q(\pi_q M \underset{R}{\otimes} N)).\end{aligned}$$

In order to see that the filtered object $AH_R(M, N)$ is complete, whenever N is bounded below, note that there exists n such that $\Sigma^n N$ is connective. In particular, for every $k \in \mathbb{Z}$ we have that

$$(\tau_{\geqslant k} M) \underset{R}{\otimes} \Sigma^n N \simeq \Sigma^n(\tau_{\geqslant k} M \underset{R}{\otimes} N)$$

is k-connective, and hence $\Sigma^{n,0}\mathrm{AH}_R(M, N)$ is connective in $\mathcal{C}^{\mathrm{fil}}$ and thus complete. It follows that $\mathrm{AH}_R(M, N)$ is complete, since complete objects in $\mathcal{C}^{\mathrm{fil}}$ form a stable subcategory. $\square$

Example A.1.17. Let $R \in \mathrm{CAlg}(\mathcal{S}p)$ be a connective ring spectrum, if M and N are R-module spectra, then the E_1-page of the Atiyah–Hirzebruch spectral sequence of the triple (R, M, N) agrees as a bigraded abelian group with the "Atiyah–Hirzebruch spectral sequence" constructed in [Elm+97b, Theorem 3.7].

Construction A.1.18. Let $\mathcal{C}$ be as above. Since $\mathcal{C}_{\geqslant 0}$ is separated and $R \in \mathrm{CAlg}(\mathcal{C}^{\heartsuit})$ is a discrete commutative algebra in $\mathcal{C}$, by [Lurb, Theorem C.5.4.9] restricting to the hearts provides an equivalence

$$\mathrm{Fun}^{L,\mathrm{lex}}(D(\mathrm{Mod}_R(\mathcal{C}^{\heartsuit})_{\geqslant 0}), \mathrm{Mod}_R(\mathcal{C})_{\geqslant 0}) \xrightarrow{\sim} \mathrm{Fun}^{L,\mathrm{ex}}(\mathrm{Mod}_R(\mathcal{C}^{\heartsuit}), \mathrm{Mod}_R(\mathcal{C}^{\heartsuit})).$$

Here, the decorations mean the full subcategory spanned by colimit preserving left exact functors and colimit preserving exact functors respectively. The identity induces a canonical map $H\colon D(\mathrm{Mod}_R(\mathcal{C}^\heartsuit))_{\geqslant 0} \to \mathrm{Mod}_R(\mathcal{C})_{\geqslant 0}$, such that the triangle

$$\begin{tikzcd}
\mathrm{Mod}_R(\mathcal{C}^\heartsuit) \arrow[rr] \arrow[dr] & & D(\mathrm{Mod}_R(\mathcal{C}^\heartsuit))_{\geqslant 0} \arrow[dl] \\
& \mathrm{Mod}_R(\mathcal{C})_{\geqslant 0} &
\end{tikzcd}$$

commutes. Furthermore, if M is a flat R-module[1], then we obtain a colimit preserving left exact functor $M \otimes_R^{\mathbb{L}} -\colon D(\mathrm{Mod}_R(\mathcal{C}^\heartsuit))_{\geqslant 0} \to \mathrm{Mod}_R(\mathcal{C})_{\geqslant 0}$. Note that, a priori this functor is different from the functor $M \otimes_R H(-)$.

Definition A.1.19. Let $\mathcal{C}$ be as above. We say that $\mathcal{C}$ satisfies the *homological flatness hypothesis* if for every discrete commutative algebra $R \in \mathrm{CAlg}(\mathcal{C}^\heartsuit)$, and choice of R-modules $M, N \in \mathrm{Mod}_R(\mathcal{C})_{\geqslant 0}$ such that M is flat, the following are equivalent:

1. The homotopy groups $\pi_k(M \otimes_R N)$ vanish for all $k > 0$.[2]

2. The homotopy groups $\pi_k(M \otimes_R^{\mathbb{L}} N)$ vanish for all $k > 0$.

We say that $\mathcal{C}$ satisfies the *weak homological flatness hypothesis* if for every discrete commutative algebra $R \in \mathrm{CAlg}(\mathcal{C}^\heartsuit)$ and flat R-module M, the functor

$$M \underset{R}{\otimes} -\colon\ \mathrm{Mod}_R(\mathcal{C}) \to \mathrm{Mod}_R(\mathcal{C})$$

is t-exact.

[1]Note, that since R is discrete there is no difference between being t-structure flat in $\mathrm{Mod}_R(\mathcal{C})$ and being flat in $\mathrm{Mod}_R(\mathcal{C}^\heartsuit)$.

[2]Here $M \otimes_R N$ is the tensor product in $\mathcal{C}_{\geqslant 0}$, and we have omitted the Eilenberg-MacLane functor, coming from the inclusion of the heart, from the notation.

Remark A.1.20. The homological flatness hypothesis might seem obtuse to the reader. We do not intend for this condition to become standard; we only provide a minimal condition that makes the proofs work. It is however often satisfied, for instances when $\mathcal{C}_{\geqslant 0}$ is *projectively rigid*[1], as shown by Liang in [Lia26, Definition 3.12]. For instance $\mathcal{S}p^{\mathrm{fil}}_{\geqslant 0}$ is projectively rigid.

Remark A.1.21. If $\mathcal{C}$ satisfies the weak homological flatness hypothesis, then we deduce that for every discrete commutative algebra R and flat R-module M, the functors $M \otimes_R -$ and $M \otimes^{\mathbb{L}}_R -$ agree. This is because they both are left exact and agree on the hearts.

Remark A.1.22. It is reasonable to ask for a $\mathbb{E}_1$-*homological flatness hypothesis*, where one replaces commutative algebra objects with associative algebra objects and modules with left and right modules. The following conjecture is true in the case where $\mathcal{C}$ satisfies the $\mathbb{E}_1$-homological flatness conjecture.

Conjecture A.1.23. If $\mathcal{C}$ satisfies the homological flatness hypothesis, then $\mathcal{C}^{\mathrm{gr}}$ satisfies the homological flatness hypothesis.

Proposition A.1.24. *Suppose that $\mathcal{C}^{\mathrm{gr}}$ satisfies the weak homological flatness hypothesis. Let R be a connective commutative algebra in $\mathcal{C}$, and let M and N be R-modules. If M is flat, then*

$$\Sigma^*\pi_*M \underset{\Sigma^*\pi_*R}{\otimes} \Sigma^*\pi_*N$$

is discrete in $\mathcal{C}^{\mathrm{gr}}$. Furthermore, we obtain an equivalence

$$\Sigma^*\pi_*M \underset{\Sigma^*\pi_*R}{\otimes} \Sigma^*\pi_*N \simeq \Sigma^*(\pi_0 M \underset{\pi_0 R}{\otimes} \pi_*N).$$

[1] A definition which, as far as we know, is due to Jiachen Liang.

Proof. It follows from Proposition A.1.9 that $\Sigma^*\pi_*M$ is flat in $\mathcal{C}^{\mathrm{gr}}$. Since $\Sigma^*\pi_*R$, $\Sigma^*\pi_*M$ and $\Sigma^*\pi_*N$ are discrete, it follows from the weak homological flatness hypothesis that so is $\Sigma^*\pi_*M \otimes_{\Sigma^*\pi_*R} \Sigma^*\pi_*N$. It follows that

$$\begin{aligned}
\Sigma^*\pi_*M \underset{\Sigma^*\pi_*R}{\otimes} \Sigma^*\pi_*N &\simeq \Sigma^*(\pi_*M \underset{\pi_*R}{\otimes} \pi_*N) \\
&\simeq \Sigma^*(\pi_0M \underset{\pi_0R}{\otimes} \pi_*R \underset{\pi_*R}{\otimes} \pi_*N) \\
&\simeq \Sigma^*(\pi_0M \underset{\pi_0R}{\otimes} \pi_*N). \qquad \square
\end{aligned}$$

Corollary A.1.25. *Suppose $\mathcal{C}^{\mathrm{gr}}$ satisfies the weak homological flatness hypothesis, and let $R \in \mathrm{CAlg}(\mathcal{C})$ be any commutative algebra object and let M be a flat R-module. If N is any R-module, then we have an equivalence*

$$\pi_*(M \underset{R}{\otimes} N) \cong \pi_*M \underset{\pi_*R}{\otimes} \pi_*N$$

in $\mathcal{C}^{\heartsuit}$.

Proof. It follows from Proposition A.1.24 that E_1-page for the Künneth spectral sequence computing $\pi_*(M \otimes_R N)$ is given by

$$\begin{aligned}
\pi_{p,q}(\Sigma^*\pi_*M \underset{\Sigma^*\pi_*R}{\otimes} \Sigma^*\pi_*N) &\cong \pi_{p,q}(\Sigma^*(\pi_0M \underset{\pi_0R}{\otimes} \pi_*N)) \\
&\cong \pi_p(\Sigma^q(\pi_0M \underset{\pi_0R}{\otimes} \pi_qN)),
\end{aligned}$$

which is concentrated on the diagonal $p = q$. In particular, the spectral sequence degenerates immediately and the result follows. $\square$

We will need a few more tools from homotopy theory.

Definition A.1.26. The category $\mathcal{C}$ has *Milnor* $\lim^1$*-sequences* if $\pi_k\colon \mathcal{C} \to \mathcal{C}^{\heartsuit}$ preserves countable products for every k.

Proposition A.1.27. *If $\mathcal{C}$ has Milnor* $\lim^1$ *sequences, then for all $X\colon \mathbb{Z}_{\geqslant 0} \to \mathcal{C}$ and $k \in \mathbb{Z}$ there is a short exact sequence*

$$0 \to \lim^1 \pi_{k+1} X \to \pi_k \lim_n X_n \to \lim_n \pi_k X_n \to 0$$

in $\mathcal{C}^\heartsuit$. Furthermore, if $\{\pi_{k+1} X_n\}_{n \geqslant 1}$ is Mittag-Leffler, then $\lim^1 \pi_{k+1} X \simeq 0$.

Proof. There is a fiber sequence

$$\lim_n X_n \to \prod_{k \in \mathbb{Z}_{\geqslant 0}} X_k \xrightarrow{d} \prod_{k \in \mathbb{Z}_{\geqslant 0}} X_k$$

where $d = \mathrm{shift} - \mathrm{id}$. In particular, splicing the long exact sequence in homotopy we get a short exact sequence

$$0 \to \operatorname{coker}(\pi_{k+1} d) \to \pi_k(\lim_n X_n) \to \ker(\pi_k d) \to 0.$$

It is a standard exercise to show that $\ker(\pi_k d) \cong \lim_n \pi_k X_n$. We define the $\lim^1$-term as

$$\lim^1 \pi_{k+1}(X_n) \coloneqq \operatorname{coker}(\pi_{k+1} d).$$

From this description it is clear that if $\{\pi_{k+1} X_n\}_{n \geqslant 0}$ is Mittag-Leffler, then $\lim^1 \pi_{k+1} X_n \cong 0$. $\square$

Definition A.1.28. A cosimplicial object $X\colon \mathbb{\Delta} \to \mathcal{C}$ is *uniformly bounded* if there exists $a \leqslant b \in \mathbb{Z}$, such that $X_n \in \mathcal{C}_{[a,b]}$ for all n. We say that X has bounds $a \leqslant b$.

Definition A.1.29. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor of presentable stable categories with t-structures. We say that F has *t-amplitude $\leqslant n$* if for all $X \in \mathcal{C}^\heartsuit$ we have $F(X) \in \mathcal{D}_{\leqslant n}$. Similarly, we say that F has *t-amplitude $\geqslant n$* if for all $X \in \mathcal{C}^\heartsuit$, we have $F(X) \in \mathcal{D}_{\geqslant n}$. We say that F has *bounded t-amplitude* if there exists $m, n \in \mathbb{Z}$ such that F has t-amplitude $\leqslant n$ and $\geqslant m$.

Definition A.1.30. An object $X \in \mathcal{C}$ has *bounded* Tor*-amplitude* if $X \otimes -\colon \mathcal{C} \to \mathcal{C}$ has bounded t-amplitude.

Example A.1.31. Any abelian group has bounded Tor-amplitude, when considered as an object in $D(\mathbb{Z})$. In fact, since the projective dimension of $\mathbb{Z}$ is 1, any abelian group has Tor-amplitude in $[0, 1]$. Furthermore, any flat abelian group has Tor-amplitude 0.

Remark A.1.32. Since $\mathcal{C}_{\geqslant 0}$ is closed under the symmetric monoidal structure any connective object $X \in \mathcal{C}_{\geqslant 0}$ has Tor-amplitude $\geqslant 0$.

Lemma A.1.33. *Let $\mathcal{C}$ be a presentable symmetric monoidal stable category with a t-structure, which is compatible with the symmetric monoidal structure. If $M\colon \mathbb{\Delta} \to \mathcal{C}$ is a cosimplicial object, which is uniformly bounded with bounds $a \leqslant b$, then the following hold:*

1. *For every $n \geqslant 0$, the partial totalizations $\operatorname{Tot}^n M_\bullet := \lim_{[k]\in\mathbb{\Delta}_{\leqslant n}} M_k$ has bounded homotopy.*

2. *For every $n \geqslant 0$ and $0 \leqslant m < n$ the fiber $F^n_m := \operatorname{fib}(\operatorname{Tot}^n M \to \operatorname{Tot}^m M)$ is $(b - m - 1)$-coconnective.*

3. *The system $\{\pi_k \operatorname{Tot}^n M_\bullet\}_{n\geqslant 0}$ is Mittag–Leffler for every $k \in \mathbb{Z}$. In fact, for every $k \in \mathbb{Z}$ there exists an $n \geqslant 0$ such that $\pi_k \operatorname{Tot}^{n+m} M_\bullet \to \pi_k \operatorname{Tot}^n M_\bullet$ is an equivalence for every $m \geqslant 1$.*

4. *If $N \in \mathcal{C}$ has bounded* Tor*-amplitude and $\mathcal{C}$ has Milnor* $\lim^1$*-sequences, then the canonical comparison map*

$$(\lim_{[k]\in\mathbb{\Delta}} M_k) \otimes N \to \lim_{[k]\in\mathbb{\Delta}} (M_k \otimes N)$$

   is an equivalence.

Proof. For (1), note that $\Delta_{\leqslant n}$ is a finite ∞-category for all n. In particular, $\mathrm{Tot}^n M_\bullet$ is a finite limit. Hence, it suffices to see that the full subcategory $\mathcal{C}^b$ spanned by bounded objects in $\mathcal{C}$ is a stable subcategory. To see this, it suffices to see that $\mathcal{C}^b$ is closed under products, fiber sequences and suspensions, all of which can be verified by using the long exact sequence in homotopy.

We now prove (2). We will assume without loss of generality that M_k is coconnective, that is $b = 0$. Let $M_{\leqslant n}$ denote the restriction of M to $\Delta_{\leqslant n}$ and let $M^n_{\leqslant n-1}$ denote the right Kan extension of $M_{\leqslant n-1}$ along $\Delta_{\leqslant n-1} \hookrightarrow \Delta_{\leqslant n}$. In this situation we have equivalences

$$\lim_{\Delta_{\leqslant n}} M_{\leqslant n} \simeq \mathrm{Tot}^n M \qquad \text{and} \qquad \lim_{\Delta_{\leqslant n}} M^n_{\leqslant n-1} \simeq \mathrm{Tot}^{n-1} M.$$

Furthermore, since limits commute with limits we see that

$$F^n_{n-1} \simeq \lim_{\Delta_{\leqslant n}} \mathrm{fib}(M_{\leqslant n} \to M^n_{\leqslant n-1}).$$

Since $\Delta_{\leqslant n-1} \to \Delta_{\leqslant n}$ is fully faithful, we see that

$$(\mathrm{fib}(M_{\leqslant n} \to M^n_{\leqslant n-1}))_k \simeq \begin{cases} 0 & k \leqslant n-1 \\ \mathrm{fib}(X_n \to (X^n_{\leqslant n-1})_n) & k = n. \end{cases}$$

In particular, by [Lur17b, Cor. 1.2.4.18] we obtain a canonical equivalence $F^n_{n-1} \simeq \Omega^n(\mathrm{fib}(X_n \to (X^n_{\leqslant n-1})_n))$. Thus to see that F^n_{n-1} is $(-n)$-coconnective it suffices to see that $\mathrm{fib}(X_n \to (X^n_{\leqslant n-1})_n)$ is coconnective. This is clear since coconnective objects are closed under limits and X_k is coconnective for all $[k] \in \Delta$.[1]

It remains to be checked that F^n_m is $(-m-1)$-coconnective for all $0 \leqslant m < n$. This can be seen inductively, by using the fiber sequence

$$F^n_{n-1} \to F^n_m \to F^{n-1}_m$$

[1]One verifies that $(X^n_{\leqslant n-1})_n)$ is coconnective by the pointwise formula for right Kan extensions, which expresses it as a limit taking values in coconnective objects.

since F_m^{n-1} is $(-m-1)$-coconnective by our inductive hypothesis and F_{n-1}^n is $(-n)$-coconnective by the above, and hence, since $m \leqslant n-1$, we see that F_m^n is $(-m-1)$-coconnective.

We now prove (3). By (2) the map $\pi_k \operatorname{Tot}^n M \to \pi_k \operatorname{Tot}^m M$ is injective, when $k = b - m$ and an isomorphism for $k \geqslant b - m$. In particular, for all k, we see that $\pi_k \operatorname{Tot}^{n+k+b} M \to \pi_k \operatorname{Tot}^{k-1+b} M$ is an equivalence proving the claim. In particular, $\{\pi_k \operatorname{Tot}^n M\}_{n\geqslant 0}$ is Mittag–Leffler for all $k \in \mathbb{Z}$.

We now prove (4). It follows from (3) that $\pi_k(\lim_{[k]\in\Delta} M_n) \cong \lim_n \pi_k(\operatorname{Tot}^n M_\bullet)$. Now, since N has bounded Tor-amplitude, then by (3) we see that $\{\pi_k(\operatorname{Tot}^n M_\bullet \otimes N)\}_{n\geqslant 0}$ is again Mittag–Leffler. It follows that

$$\begin{aligned} \pi_k((\lim_{[n]\in\Delta} M_n) \otimes N) &\simeq \lim_n \pi_k((\operatorname{Tot}^n M_\bullet) \otimes N) \\ &\simeq \lim_n \pi_k(\operatorname{Tot}^n(M_\bullet \otimes N)) \\ &\simeq \pi_k(\lim_{[n]\in\Delta} (M_n \otimes N)). \end{aligned}$$

Here we use that $- \otimes N$ preserves finite limits. □

Remark A.1.34. Part (2) and (3) in Lemma A.1.33 are completely contained in [Lur17b, Prop. 1.2.4.5]. We only include a proof here for completeness' sake, and to illustrate that if you only care about connectivity bounds, then you get by with a slightly simpler proof.

A.2 Stacks on a site

In this section we introduce the category of stacks on a (large) site and show that this satisfies many of the good categorical properties the ∞-topos of sheaves on a small site does.

Definition A.2.1 ([BDL25, Definition 2.1.1.5])**.** If $(\mathcal{C}, \tau)$ is a site, then we define the *category of stacks* on $\mathcal{C}$ to be the colimit

$$\mathrm{Stk}(\mathcal{C}) \coloneqq \mathrm{colim}_{\mathcal{C}_0 \subseteq \mathcal{C}} \mathrm{Shv}_\tau(\mathcal{C}),$$

indexed over all small subsites $(\mathcal{C}_0, \tau)$ of $(\mathcal{C}, \tau)$.

Convention A.2.2. For the rest of this section we assume that any Grothendieck topology we consider is subcanonical, that is the Yoneda embedding $よ\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ factors through τ-sheaves. Furthermore, we assume that $\mathcal{C}$ has finite limits. We will also suppress the Grothendieck topology from our notation, except if multiple sites are considered.

Remark A.2.3 ([BDL25, Eq. 2.1.1.8])**.** For every site $\mathcal{C}$ the Yoneda embedding on every small subsite $\mathcal{C}_0$ factors through the category of sheaves on $\mathcal{C}_0$, so we obtain a functor

$$\mathrm{Spec}\colon \mathcal{C} \to \mathrm{Stk}(\mathcal{C}).$$

Definition A.2.4 ([BDL25, Definition 2.1.1.9])**.** A stack $X \in \mathrm{Stk}(\mathcal{C})$ is affine, if it is in the essential image of Spec.

Proposition A.2.5 ([BDL25, Prop. 2.1.2.1])**.** *For a site $\mathcal{C}$ the category of stacks on $\mathcal{C}$ has all small colimits and finite limits.*

Proposition A.2.6 ([BDL25, Prop. 2.1.2.2])**.** *For any category $\mathcal{D}$ with small limits, restriction along* $\mathrm{Spec}\colon \mathcal{C} \to \mathrm{Stk}(\mathcal{C})$ *induces a fully faithful functor*

$$\mathrm{Fun}^R(\mathrm{Stk}(\mathcal{C})^{\mathrm{op}}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

with essential image given by the functors $\mathcal{C} \to \mathcal{D}$, which are sheaves.

Definition A.2.7 ([BDL25, Definition 2.1.1.2])**.** A *map of sites* $(\mathcal{C}, \tau) \to (\mathcal{C}', \tau')$ consists of a functor $\mathcal{C} \to \mathcal{C}'$, which preserves covering families and pullbacks along coverings. A map of sites is *exact* if it preserves all finite limits.

Proposition A.2.8 ([BDL25, Prop. 2.1.2.6])**.** *Any map of sites* $f: (\mathcal{C}, \tau) \to (\mathcal{C}', \tau')$ *induces a colimit preserving functor*

$$f_!: \operatorname{Stk}(\mathcal{C}) \to \operatorname{Stk}(\mathcal{C}').$$

If f is exact, then $f_!$ preserves finite limits.

Definition A.2.9. A presheaf $X: \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is *small* if it is a small colimit of representable functors. We denote the category of small presheaves on $\mathcal{C}$ by $\mathcal{P}(\mathcal{C})$[1].

Remark A.2.10. If $\mathcal{C}$ is a coaccessible category with the trivial Grothendieck topology, then the category of stacks on $\mathcal{C}$ is the category of small presheaves on $\mathcal{C}$.

Corollary A.2.11 ([BDL25, Cor. 2.1.2.7])**.** *For any site $\mathcal{C}$ there exists a left exact colimit preserving functor*

$$L: \mathcal{P}(\mathcal{C}) \to \operatorname{Stk}(\mathcal{C}).$$

Remark A.2.12. One might believe that not every stack can be written as a colimit of representables, since for any small subsite $\mathcal{C}_0 \subseteq \mathcal{C}$ the inclusion $\operatorname{Shv}(\mathcal{C}_0) \to \operatorname{Stk}(\mathcal{C})$ might not preserve colimits. However, if $\mathcal{C}$ is coaccessible, then the statement still holds. This can be seen as follows. Let $X \in \operatorname{Stk}(\mathcal{C})$ be any stack on $\mathcal{C}$. By construction, there exists a small subsite $\mathcal{C}_1$, such that X is in the image of

$$\iota_1: \operatorname{Shv}(\mathcal{C}_1) \hookrightarrow \operatorname{Stk}(\mathcal{C}).$$

[1]This is only a mild abuse of notation, since if $\mathcal{C}$ is small, then $\mathcal{P}(\mathcal{C})$ agrees with the category of presheaves on $\mathcal{C}$.

In particular, there is an equivalence $\operatorname{colim}_{よ(y)\to X} よ(y) \simeq X$ in $\mathrm{Shv}(\mathcal{C}_1)$. We claim this colimit is preserved by ι_1. To see this, consider the diagram:

$$\begin{tikzcd}
\mathcal{P}(\mathcal{C}_1) \arrow[r, "L_1"] \arrow[d, "i_1"] & \mathrm{Shv}(\mathcal{C}_1) \arrow[d, "\iota_1"] \\
\mathcal{P}(\mathcal{C}) \arrow[r, "L"] & \mathrm{Stk}(\mathcal{C}).
\end{tikzcd}$$

Since $\mathcal{C}_1$ is small, L_1 is left adjoint to the inclusion from sheaves to presheaves. In particular, we obtain an equivalence

$$\operatorname{colim}_{よ(y)\to X} よ(y) \xrightarrow{\sim} X,$$

inducing the above equivalence in $\mathrm{Shv}(\mathcal{C}_1)$. Now, by [BDL25, Cor. 2.1.2.7] the functor $L\colon \mathcal{P}(\mathcal{C}) \to \mathrm{Stk}(\mathcal{C})$ preserves colimits, so it suffices to show that the above colimit is preserved by i_1, which follows from the fact that $\mathcal{C}$ is coaccessible.

Proposition A.2.13 ([BDL25, Prop. 2.1.2.9])**.** *If $\mathcal{C}$ is a site, then the following properties hold:*

1. *Coproducts in $\mathrm{Stk}(\mathcal{C})$ are disjoint.*
2. *Colimits in $\mathrm{Stk}(\mathcal{C})$ are universal.*
3. *Groupoids in $\mathrm{Stk}(\mathcal{C})$ are effective.*
4. *The functor $\mathrm{Stk}(\mathcal{C})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}$ sending X to $\mathrm{Stk}(\mathcal{C})_{/X}$ preserves limits.*

Definition A.2.14 ([BDL25, Definition 2.1.3.1])**.** A map $f^*\colon \mathcal{C} \to \mathcal{D}$ in $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ is called *affine*, if the right adjoint $f_*\colon \mathcal{D} \to \mathcal{C}$ induces a symmetric monoidal equivalence

$$\mathrm{Mod}_{f_*1_{\mathcal{D}}}(\mathcal{C}) \xrightarrow{\sim} \mathcal{D}.$$

Definition A.2.15 ([BDL25, Definition 2.1.3.2])**.** A *quasi-coherent sheaf context* is a triple $(\mathcal{C}, \tau, Q)$, consisting of a site $(\mathcal{C}, \tau)$ and a τ-sheaf $Q\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ of presentably symmetric monoidal stable categories subject to the following conditions.

1. For every map $f\colon X \to Y$ in $\mathcal{C}$ the induced functor $Q(f)\colon Q(Y) \to Q(X)$ is affine.

2. The functor Q preserves pushout squares.

Definition A.2.16 ([BDL25])**.** If $(\mathcal{C}, \tau, Q)$ is a quasi-coherent sheaf context, then we let

$$\mathrm{QCoh}\colon \mathrm{Stk}(\mathcal{C})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$$

denote the canonical limit preserving extension guaranteed by [BDL25, Prop. 2.1.2.2].

A.3 Dirac Geometry

We will now review the needed background on Dirac Geometry following Hesselholt–Pstrągowski. This is a version of algebraic geometry where the test objects are so-called "*Dirac rings*", that is $\mathbb{Z}$-graded rings which are graded commutative in the sense that if x and y are homogeneous elements, then

$$xy = (-1)^{|x||y|}yx.$$

Here $|x|$ denotes the degree of x. Formally the category of Dirac rings can be defined as

$$\mathrm{DRing} \coloneqq \mathrm{CAlg}((\mathcal{S}p^{\mathrm{gr}})^{\heartsuit})$$

where $\mathcal{S}p^{\mathrm{gr}}$ is given the diagonal t-structure of Proposition III.2.1.20. This follows from the identification $(\mathcal{S}p^{\mathrm{gr}})^{\heartsuit} \simeq \mathrm{Ab}^{\mathrm{gr}}$ which can be made symmetric monoidal by equipping the right-hand side with the Koszul graded tensor product.

Definition A.3.1 ([HP24])**.** We define the category of *affine Dirac stacks* to be $\mathrm{DAff} := \mathrm{DRing}^{\mathrm{op}}$. A collection of maps $\{R_i \to R\}_{i \in I}$ is a *flat cover* in DAff if it contains a finite collection of maps, such that

$$R \to \prod_{j=1}^{n} R_j$$

is a faithfully flat map of Dirac rings.

Theorem A.3.2 ([HP24, Prop. 2.4, Lemma 2.10])**.** The collection of flat covers on DAff defines a finitary Grothendieck topology making DAff into an accessible site.

Definition A.3.3. A functor $X\colon \mathrm{DRing} \to \mathcal{S}$ is a *Dirac stack* if X is a sheaf for the flat topology on DAff and X is accessible. We denote the category of Dirac stacks by DStk.

Corollary A.3.4. *There is a canonical equivalence*

$$\mathrm{Stk}(\mathrm{DAff}) \xrightarrow{\sim} \mathrm{DStk}$$

between the category of stacks on DAff *and the category of Dirac stacks.*

Proof. This follows immediately from the fact that DAff is an accessible site and [BDL25, Cor. 2.1.2.5]. □

Theorem A.3.5 ([HP25, Prop. 3.3, Addendum 3.4][HP24, Prop. 3.23])**.** If

$$D\colon \mathrm{DRing} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$$

denotes the functor sending a Dirac ring R to its category of graded module spectra $\mathrm{Mod}_R(\mathcal{S}p^{\mathrm{gr}})$[1], then the triple $(\mathrm{DAff}, \mathrm{flat}, D)$ is a quasi-coherent sheaf context.

[1]This is seen in [HP24, Prop. 3.4] to be canonically equivalent to the derived category $D(\mathrm{Mod}_R(\mathrm{Ab}^{\mathrm{gr}}))$ as objects of $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$.

Definition A.3.6. A Dirac stack X is geometric if it admits a flat cover[1] by an affine Dirac scheme.

Remark A.3.7. The definition of geometric Dirac stack in Definition A.3.6 is less general than the one given in [HP24, Section 2.3], where the authors only require that the map be representable by flat maps of Dirac schemes. For the purposes of this paper the above definition will suffice.

Remark A.3.8. In this paper, we will concern ourselves with stacks on multiple different sites. Specifically the sites $\mathrm{Aff}, \mathrm{FAff}$ and DAff with the π_*-flat, flat and flat topologies respectively will be of interest. We will see that these all refine to quasi-coherent sheaf contexts and will always abuse notation and write QCoh for the category of quasi-coherent sheaves on a stack on any of these sites. This is mostly harmless, since the correct version can always be deduced by type-checking.

We will need the following result on t-structures on the category of quasi-coherent sheaves on a geometric Dirac stack. This is [HP24, Theorem 3.27] specialized to the case of a geometric stack in the sense of Definition A.3.6.

Theorem A.3.9 ([HP24, Theorem 3.27])**.** If X is a geometric Dirac stack, then $\mathrm{QCoh}(X)$ admits a t-structure $(\mathrm{QCoh}(X)_{\geqslant 0}, \mathrm{QCoh}(X)_{\leqslant 0})$ such that the following hold.

1. The t-structure is left and right complete, compatible with the symmetric monoidal structure and compatible with filtered colimits.

[1] Here flat cover means and epimorphism, which is representable by affines and the map induced by base-change corresponds to a flat map of Dirac rings.

2. The category $\mathrm{QCoh}(X)_{\geqslant 0}$ is complete Grothendieck prestable and the functor

$$\mathrm{QCoh}(X)_{\geqslant 0} \hookrightarrow \mathrm{QCoh}(X)$$

identifies $\mathrm{QCoh}(X)$ with the stabilization of $\mathrm{QCoh}(X)_{\geqslant 0}$.

3. A quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(X)$ is (co)connective if and only if for every flat cover $p\colon \operatorname{Spec} R \to X$ by an affine $p^*\mathcal{F}$ is (co)connective.

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